On a Local-Global Property of Algebraic Dynamics

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Outline

Notations.

K	a number field.
M_K	the set of inequivalent places of K.
M_K^∞	the set of archimedean places of K.
$\mathfrak{p}_v$	the prime ideal associated to a finite place $v \in M_K$.
K_v	the completion of K with respect to $v \in M_K$,
A_K	the ring of adèles of K.
X/K	an projective variety defined over K,
$X(F)$	the set of points of X defined over the field F.

From global to local

We have

$$X(K) \hookrightarrow X(K_v) \quad \text{for all } v \in M_K.$$

Suppose $X(K) \neq \emptyset$, then

$$X(K_v) \neq \emptyset \qquad \forall v \in M_K \quad \text{(The Hasse condition).}$$

Remark

It's always true that $X(K_v) \neq \emptyset$ for all but finitely many $v \in M_K$. So, one only has to check whether or not $X(K_v) \neq \emptyset$ for the remaining finite subset of M_K and in many cases this can be done in a finite steps of computations.

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From local to global

Question

Is the Hasse condition a sufficient condition to imply $X(K) \neq \emptyset$?

If the Hasse condition is a sufficient condition for $X(K) \neq \emptyset$ then X/K is said to satisfy the *Hasse principle*.

Example (The Hasse-Minkowski Theorem)

$F(x_1, \ldots, x_n)$: a quadratic form defined over K where $n \geq 2$.

$$\begin{aligned} F(x_1, \ldots, x_n) = \; & 0 \quad \text{has nontrivial solution } (x_1, \ldots, x_n) \in K^n \\ & \Updownarrow \\ F(x_1, \ldots, x_n) = \; & 0 \quad \text{has nontrivial solution } (x_1, \ldots, x_n) \in K_v^n \\ & \quad \text{for all } v \in M_K. \end{aligned}$$

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Counter-example to the Hasse principle

Example (C.-E. Lind)

Let $X/\mathbb{Q}$ be the curve defined by equation

$$2y^2 = x^4 - 17.$$

Then,

$$X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but } X(\mathbb{Q}) = \emptyset.$$

Example (S. Selmer)

Let $X/\mathbb{Q}$ be the cubic curve defined by equation

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Putting local information together

Question

What is the obstruction for X to satisfy the Hasse principle ?

$$X(\mathbb{A}_K) := \prod_{v \in M_K} X(K_v) \quad \text{(the set of adèlic points of } X\text{)}.$$

- Since $X(K_v)$ is compact with respect to the topology induced by that of K_v, it follows that $X(\mathbb{A}_K)$ is a compact topological space with respect to the product of topologies on $X(K_v)$ for all $v \in M_K$.
- We have

$$X(K) \hookrightarrow X(\mathbb{A}_K)$$
$$P \mapsto (P)_{v \in M_K}$$

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$$\begin{aligned} X(K) &\hookrightarrow X(\mathbb{A}_K) \\ P &\mapsto (P)_{v \in M_K} \end{aligned}$$

The Brauer set

Question

How to cut out the global information $X(K)$ from $X(\mathbb{A}_K)$?

Remark (Y. Manin)

There is a closed subset $X(\mathbb{A}_K)^{\mathrm{Br}}$ of $X(\mathbb{A}_K)$ defined by certain conditions related to the Brauer group $\mathrm{Br}(X)$ such that

$$X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K).$$

Theorem (Y. Manin)

X/K : projective, smooth curve with $g(X) = 1$. $\mathcal{E} = \mathrm{Jac}(X)$. Suppose that the Tate-Shafarevich group $\text{Ш}(\mathcal{E})$ is finite. If $X(K) = \emptyset$ and $X(\mathbb{A}_K) \neq \emptyset$, then $X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$.

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Capturing the Brauer set

Let X/K be a smooth, projective curve defined over K.
Let $A = \mathrm{Jac}(X)(= \mathrm{Pic}^0(X))$, the Jacobian of the curve X and a fixed jacobian embedding

$$\jmath : X \hookrightarrow A$$

which assumed to be defined over K.

$A(K_v)^0$	the identity component of $A(K_v)$ for $v \in M_K^\infty$ and 0 for $v \in M_K^0$.
$A(\mathbb{A}_k)_\bullet$	$= \prod_{v \in M_K} A(K_v)/A(K_v)^0$.
$X(\mathbb{A}_k)_\bullet$	the image of $\jmath(X(\mathbb{A}_K))$ under the projection $A(\mathbb{A}_K) \to A(\mathbb{A}_k)_\bullet$.

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Adèlic closure of Mordell-Weil Groups

$\overline{A(K)}$: the topological closure of $A(K)$ inside $A(\mathbb{A}_k)_\bullet$

Theorem

(**V. Sharaschkin**) *Suppose that* $\text{Ш}(A)$ *is finite. Then,*

$$X(K) \simeq X(\mathbb{A}_k)_\bullet \cap A(K) \subseteq X(\mathbb{A}_k)_\bullet \cap \overline{A(K)} \simeq X(\mathbb{A}_K)^{\mathrm{Br}}.$$

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Function field analogue.

F a finitely generated extension and of transcendence degree 1 over a field k,

A an abelian variety over F,

X a closed F-subscheme of A.

$X(\mathrm{A}_k)_\bullet, A(\mathrm{A}_k)_\bullet$ defined as above.

Theorem

(B. Poonen and F. Voloch)
(1) If Char $k = 0$ *then* $X(F) = X(\mathrm{A}) \cap \overline{A(F)}$.
(2) If Char $k = p > 0$ *then, under certain mild conditions,* $X(F) = X(\mathrm{A}_k)_\bullet \cap \overline{A(F)}$.

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Orbits, (pre)periodic points

Let X be any algebraic variety over K and let

$$\varphi : X \to X \quad \text{of degree } d \geq 2 \text{ over } K.$$

Dynamical system associated to φ on X is roughly the classification of points in X under the action of φ via iterations $\{\varphi^n \mid n \geq 0\}$ Let $P \in X(K)$. The (forward) orbit of P

$$\mathcal{O}_\varphi(P) := \{\varphi^n(P) \mid n = 0, 1, 2, \ldots\}.$$

P is called *preperiodic* if $\sharp\mathcal{O}_\varphi(P) < \infty$; *wandering* if $\sharp\,\mathcal{O}_\varphi(P) = \infty$.

Comparison: dynamical systems v.s. abelian variety

preperiodic points $\longleftrightarrow$ torsion points

wandering points $\longleftrightarrow$ point of infinite order

orbit $\mathcal{O}_\varphi(P) \longleftrightarrow$ Cyclic subgroup of $A(K)$.

Canonical height $\hat{h}_\varphi \longleftrightarrow$ Néron-Tate height on A

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A local-global question

Let V be a subvariety of X and let $P \in X(K)$ be wandering.
$\overline{\mathcal{O}_\varphi(P)}$ = the topological closure of $\mathcal{O}_\varphi(P)$ in $X(\mathbb{A}_K)$.

We have an inclusion

$$\mathcal{O}_\varphi(P) \cap V(K) \subseteq \overline{\mathcal{O}_\varphi(P)} \cap V(\mathbb{A}_K).$$

Question

Is it true that

$$\mathcal{O}_\varphi(P) \cap V(K) = \overline{\mathcal{O}_\varphi(P)} \cap V(\mathbb{A}_K)?$$

Or, when does the equality hold?

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Dynamical Brauer-Manin

Definition

A subvariety W of X is *φ-preperiodic* if there are integers $n > m$ such that $\varphi^n(W) = \varphi^m(W)$. If also $\dim(W) \geq 1$, we say that W is *nontrivial*.

Definition

Let V_φ^{pp} be the union of all nontrivial φ-preperiodic subvarieties of V. Then we say that $V(K)$ is *Brauer–Manin unobstructed* (*for* φ) if for every point $P \in X(K)$ satisfying $\mathcal{O}_\varphi(P) \cap V_\varphi^{\text{pp}} = \emptyset$ we have

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Power maps and translated tori

Theorem (L.C.Hsia and J. Silverman)

let

$$\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad \varphi([X, Y, Z]) = [X^d, Y^d, Z^d]$$

for some $d \geq 2$. *Let* $k, \ell \geq 1$, *let* $A, B \in K$, *and let* $V \subset \mathbb{P}^2$ *be the curve*

$$V : AX^k = BY^\ell.$$

Then, $V(K)$ *is Brauer-Manin unobstructed. That is, for a wandering point* $P \in \mathbb{P}^2(K)$, *one of the following two statements is true*:

(i)$\mathcal{O}_\varphi(P) \cap V(K) = \overline{\mathcal{O}_\varphi(P)} \cap V(\mathbb{A}_K)$.

(ii) *The variety* V *is preperiodic for* φ, *and if* $\mathcal{O}_\varphi(P) \cap V(K)$ *is non-empty then there exists an* $i \geq 0$ *such that* $\mathcal{O}_\varphi(P) \cap \varphi^i(V)$ *is an infinite set.*

Translated abelian subvarieties

Theorem (L.C.Hsia and J. Silverman)

Let A/K be an abelian variety, and let B/K be an abelian subvariety of A of codimension 1. Fix a $T \in A(K)$and let $V = B + T$ be the translation of B by T.
Let $d \geq 2$ and consider the multiplication-by-d map

$$[d] : A \to A.$$

Then, $V(K)$ is Brauer-Manin unobstructed. That is, for nontorsion point $P \in A(K)$, one of the following two statements is true:
(i)$\mathcal{O}_d(P) \cap V(K) = \overline{\mathcal{O}_d(P)} \cap V(\mathrm{A}_K)$.
(ii)*The variety V is $[d]$-preperiodic.*
Further, in case (ii), the point T has finite order in the quotient variety A/B.

Abelian varieties and general subvarieities

Theorem (L.C.Hsia and J. Silverman)

Let A/K be an abelian variety and let V/K be a subvariety of A. Assume that

(i) *V does not contain a translate of a positive-dimensional abelian subvariety of A.*

(ii) $V(K) = V(\mathsf{A}_k)_\bullet \cap \overline{A(K)}$, *where the closure and the equality take place in $A(\mathsf{A}_k)_\bullet$.*

Then, $V(K)$ is Brauer-Manin unobstructed for any $\varphi \in \mathrm{End}(A)$ such that $\deg(\varphi) \geq 2$ and that $\mathbb{Z}[\varphi]$ is an integral domain.

The case of power maps

Recall the d^{th}-power map

$$\varphi([X,Y,Z]) = \left[X^d, Y^d, Z^d\right]$$

and V is the curve defined by equation

$$AX^k = BY^\ell.$$

Let $P = [\alpha, \beta, \gamma] \in \mathbb{P}^2(K)$ be a wandering point for φ.

(The main case) Consider the case $\alpha\beta\gamma \neq 0$ and $AB \neq 0$

Suppose there exists

$$Q = (Q_v)_{v \in M_K} \in \overline{\mathcal{O}_\varphi(P)} \cap V(\mathbb{A}_K) \setminus \mathcal{O}_\varphi(P) \cap V(K).$$

We will show that B/A is a root of unity and hence prove that V is actually preperiodic for the power map φ.

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Adèlic closure of an orbit

Let

$$Q = (Q_v)_{v \in M_K} \in \prod_{v \in M_K} \mathbb{P}^2(K_v).$$

Then,

$$Q \in \overline{\mathcal{O}_\varphi(P)} \setminus \mathcal{O}_\varphi(P)$$

if and only if there is an infinite set of positive integers $\mathcal{N}_{P,Q} \subset \mathbb{N}$ such that

$$Q_v = \underset{\substack{n \in \mathcal{N}_{P,Q} \\ n \to \infty}}{v\text{-lim}} \varphi^n(P) \quad \forall\, v \in M_K. \tag{1}$$

• "v-lim" indicates the limit is being taken in the v-adic topology.
• $\mathcal{N}_{P,Q}$ depends only on P and Q, but it must be independent of $v \in M_K$.

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Aèlic closure

Dehomogenize so that $P = [\alpha, \beta, 1]$ and let

$$S = M_K^\infty \cup \{v \in M_K \mid |\alpha|_v \neq 1 \text{ or } |\beta|_v \neq 1\}$$

For $v \notin S$, we have

$$Q_v = \underset{\substack{n \in \mathcal{N}_{P,Q} \\ n \to \infty}}{v\text{-lim}} \varphi^n(P) = [x_v, y_v, 1] \in V(K_v)$$

It follows that

$$\underset{\substack{n \in \mathcal{N}_{P,Q} \\ n \to \infty}}{v\text{-lim}} \left(\frac{\alpha^k}{\beta^\ell}\right)^{d^n} = \frac{B}{A} \quad \forall v \in M_K \setminus S. \tag{2}$$

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The key proposition

Proposition

Let $\lambda, \xi \in K^*$. *Assume that*

- $\exists$ *a finite* $S \subset M_K$ *including all archimedean places of* K,
- $\exists$ *infinite* $\mathcal{N} \subset \mathbb{N}$,

such that

$$\xi = v\text{-}\lim_{\substack{n \in \mathcal{N} \\ n \to \infty}} \lambda^{d^n} \quad \forall v \in M_K \setminus S. \tag{3}$$

Then, both λ *and* ξ *are roots of unity.*

Bang-Zsigmondy's theorem

Theorem (Bang, Zsigmondy, Birkhoff-Vandiver, Postnikova-Schinzel)

Let K be a number field, let $\lambda \in K^*$ be an element that is not a root of unity, and let

$$S_\lambda = M_K^\infty \cup \{v \in M_K : |\lambda|_v \neq 1\}.$$

For each $v \notin S_\lambda$, let $f_v(\lambda)$ denote the order of λ in $\mathbb{F}_v^*$, the multiplicative group of residue field at v. Then the set

$$\mathbb{N} \setminus \{f_v(\lambda) : v \notin S_\lambda\}$$

is finite, i.e., all but finitely many positive integers occur as the order modulo $\mathfrak{p}$ of λ for some prime $\mathfrak{p}$ of K.

The proof of the key proposition

Recall Equation (3):

$$\xi = v\text{-}\lim_{\substack{n\in\mathcal{N}\\ n\to\infty}} \lambda^{d^n} \quad \forall v \in M_K \setminus S.$$

Assume that λ is not a root of unity.
Then, Bang-Zsigmondy's Theorem implies that

$$\exists \text{ infinite sequence } n_1, n_2, n_3, \ldots \in \mathcal{N}$$

and for each i, there exists $v_i \in M_K \setminus S$ such that

$$f_{v_i}(\lambda) = d^{n_i}.$$

We have

$$\lambda^{d^{n_i}} \equiv \lambda^{f_{v_i}(\lambda)} \equiv 1 \pmod{\mathfrak{p}_i} \qquad \text{for all } i = 1, 2, 3, \ldots.$$

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Proof - continued

For any given positive integer m, we can find m distinct places $v_1, \ldots, v_m \notin S$ so that

$$\lambda^{d^n} \equiv 1 \pmod{\mathfrak{p}_i} \quad \text{for all } 1 \leq i \leq m \text{ and all sufficiently large } n \in \mathcal{N}.$$

On the other hand, Equation (3) implies that

$$\lambda^{d^n} \equiv \xi \pmod{\mathfrak{p}_i} \qquad \text{for all sufficiently large } n \in \mathcal{N} \text{ and all } 1 \leq i \leq m. \tag{4}$$

Hence,

$$\xi \equiv 1 \pmod{\mathfrak{p}_i} \qquad \text{for all } 1 \leq i \leq m.$$

As m is arbitrary, this forces $\xi = 1$.

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Equation (3) becomes:

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Apply Bang–Zsigmondy's Theorem again to deduce that λ is a root of unity.

- The main theorem (the case of power map) follows from applying the key proposition.

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Abelian varieties and subvarieties

For the case of abelian subvariety, we apply an analogue of Bang-Zsigmondy's theorem for elliptic curves.

Theorem (Elliptic Zsigmondy Theorem)

Let E be an elliptic curve defined over K, and let $P \in E(K)$ be a nontorsion point, and let $S \subset M_K$ be a finite set of places including M_K^∞ and all places of bad reduction for E. For each place $v \notin S$, let $f_v(P)$ be the order of $P \pmod{\mathfrak{p}_v}$ in $E(\mathbb{F}_{\mathfrak{p}_v})$. Then, the set

$$\mathbb{N} \smallsetminus \{f_v(P) : v \notin S\}$$

is a finite set.

Remark

J. Silverman establishes the case for $K = \mathbb{Q}$. J. Cheon and S. Hahn prove the case for K general number field.

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Proof of Theorem on translated subvarieties

Assume that there is a

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where $V = B + T$ for some $T \in A(K)$.

As in the case of power maps, we have an infinite set $\mathcal{N}$ of positive integers so that

$$Q_v = v\text{-}\lim_{\substack{n \in \mathcal{N} \\ n \to \infty}} [d^n]P.$$

Pass to the quotient $E = A/B$ an elliptic curve over K. We have

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Proof continued

Our assumption says that

$$v\text{-}\lim_{\substack{n\in\mathcal{N}\\ n\to\infty}} [d^n](\bar{P}) = \bar{T} \quad \forall v \in M_K.$$

The remaining argument is to apply the elliptic Zsgmondy's theorem as in the case of power maps.

Question

1. Study the local-global question for the power map φ and arbitrary hypersurface V in $\mathbb{P}^N$.
2. For subvarieties of an abelian variety, can the assumption $V(K) = V(\mathrm{A}_K) \cap \overline{A(K)}$ be removed? i.e. $V(K)$ is Brauer-Manin unobstructed for arbitrary subvariety of A?
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Thank you !