$K_{2i}O_F$ for $\mathbb{Z}_p$-extension

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Outline:

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- Iwasawa's Theorem for $K_{2n}O_F$
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Iwasawa's Theorem

We recall a classical result from Iwasawa Theory.

Let F be a number field. For a prime p, let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension and let F_n be the unique intermediate field for F_∞/F such that $[F_n : F] = p^n$, $n \geq 0$. Let p^{e_n} be the exact power of p diving the class number of F_n.

Iwasawa's Theorem. There exist integers $\lambda \geq 0, \mu \geq 0$ and ν, all independent of n, and an integer n_0 such that, for all $n \geq n_0$,

$$e_n = \lambda n + \mu p^n + \nu.$$

Iwasawa's Theorem for $K_{2n}O_F$

Let F be a number field.

Assume that $\mu_p \subset F$ if $p > 2$ and $\mu_4 \subset F$ if $p = 2$.

Let M be the maximal abelian p-extension of F_∞ unramified outsider p.

Theorem. For any $i \geq 1$, there exist integers n_i and ν_i such that, for all $n \geq n_i$,

$$e(i)_n = \lambda n + \mu p^n + \nu_i,$$

where $p^{e(i)_n} = \sharp K_{2i}(O_{F_n})\{p\}$, λ and μ are the classical Iwasawa invariants of the Λ-module $\mathrm{Gal}(M/F_\infty)$ independent of i and n, and ν_i is a constant independent of n.

Remark. J. Coates, On K_2 and classical conjectures in algebraic number theory, Ann. Math., 95(1972), pp.99-116, proves the same assertion for $i = 1$.

Some Lemmas on Λ-modules

Let F be a number field with degree d.

Assume that

$\mu_p \subset F$ if $p > 2$ and $\mu_4 \subset F$ if $p = 2$.

Let q_0 be the largest power of p such that $\mu_{q_0} \subset F$.

Put $q_n = q_0 p^n$.

Write $F_n = F(\mu_{q_n})$ and $F_\infty = \bigcup_{n \geq 0}^{\infty} F_n$.

Then F_∞/F is a $\mathbb{Z}_p$-extension, and as usual, we write $\Gamma = \mathrm{Gal}(F_\infty/F)$, $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. Let

$$\kappa : \Gamma \longrightarrow 1 + q_0\mathbb{Z}_p$$

be the isomorphism determined by

$$\gamma(\zeta) = \zeta^{\kappa(\gamma)}, \qquad \text{for all } \zeta \in W = \bigcup_{n \geq 0} \mu_{p^n}, \;\; \gamma \in \Gamma.$$

Let $\Lambda = \mathbb{Z}_p[[T]]$ be the ring of formal power series in an indeterminate T with coefficients in $\mathbb{Z}_p$. Choose, once and for all, a topological generator γ_0 of Γ. Then each compact Γ-module X admits a unique structure of compact Λ-module such that

$$(1 + T)x = \gamma_0 x$$

for every x in X.

Let $\iota : \Lambda \longrightarrow \Lambda$ be the automorphism given by

$$\iota\left(\sum_{m=0}^{\infty} c_m T^m\right) = \sum_{m=0}^{\infty} c_m(\kappa(\gamma_0)/(1+T) - 1)^m.$$

Given any Λ-module Y, denote by $Y^\bullet$ the Λ-module with the same underlying group as Y but with Λ-module structure obtained from that of Y by composition with ι.

Let M be a Γ-module.

Lichtenbaum, On the values of zeta and L-functions: I, Ann., Math., 96(1972), pp.338-360, defines $M[n]$:

As $\mathbb{Z}_p$-module$M[n]$ is M;

γ action on $M[n]$ is given by the following:

For any $\gamma \in \Gamma$ and $x \in M$, $\gamma * x = \kappa(\gamma)^n \gamma(x)$.

Thus $M[n]$ is isomorphic to $M(n)$ as Γ-modules.

For any $n \in \mathbb{Z}$, we put

$$T^*_{(n)} = \kappa(\gamma_0)^n (1 + T) - 1.$$

Lemma 3.1. Let $\omega_n(T) = (1+T)^{p^n} - 1$. For any non-zero element $g(T) \in \Lambda$, let M denote the Λ-module $\Lambda/(g(T))$. And let $h: M \longrightarrow M$ be the Λ-homomorphism given by multiplication by $\omega_n(T)$.

(1) (Lichtenbaum) $M[m]$ is isomorphic to $\Lambda/(g(T^*_{(-m)}))$ as Λ-module.

(2) h has a finite cokernel if and only if $\prod_{i=0}^{n} g(\zeta_{p^i} - 1) \neq 0$, and, if $\prod_{i=0}^{n} g(\zeta_{p^i} - 1) \neq 0$, the order of the cokernel is $\prod_{i=0}^{n} |g(\zeta_{p^i} - 1)|_{v_i}^{-1}$,

where the valuation $|\cdot|_{v_i}$ is the standard valuation of the field $\mathbb{Q}_p(\zeta_{p^i})$ such that $|\zeta_{p^i}-1|_{v_i}=1/p$ for all $i\geq 1$, and $|\cdot|_{v_0}=|\cdot|_p$ on $\mathbb{Q}_p$ such that $|p|_p=1/p$.

(3) h is injective if $\prod_{i=0}^{n} g(\zeta_{p^i}-1)\neq 0$ or its kernel is infinite if $\prod_{i=0}^{n} g(\zeta_{p^i}-1)=0$.

Lemma 3.2. For all $h(T) \in \Lambda$ such that $h(T)$ and $\omega_n(T)$ are relatively prime, we have

$$\sharp \frac{\Lambda}{(\omega_n(T), h(T))} = \prod_{i=0}^{n} |h(\zeta_{p^i} - 1)|_{v_i}^{-1}.$$

Let M be a discrete Λ-module.

$\widehat{M} = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ with Λ-action given by the following formula:

For $\lambda \in \Lambda,\ \ y \in M,\ \ \varphi \in \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$,

$$(\lambda\varphi)(y) = \varphi(\lambda y).$$

Lemma 3.3. Let M be a discrete Λ-module and assume that its Pontryagin dual $\widehat{M}$ is a finitely generated torsion Λ-module with no non-trivial finite Λ-submodule, and the following sequence is exact:

$$0 \longrightarrow \widehat{M} \longrightarrow \oplus_{j=1}^{r} \Lambda/(f_j(T)) \longrightarrow D \longrightarrow 0$$

where D is a finite Λ-module. Put $f(T) = \Pi_{j=1}^{r} f_j(T)$. Then the following assertions are equivalent for all integers m and $n \geq 0$:

$$\text{(i)} \;\; M(m)^{\Gamma_n} \text{ is finite}, \quad \text{(ii)} \;\; M(m)_{\Gamma_n} = 0,$$

$$\text{(iii)} \;\; \prod_{i=0}^{n} f(\kappa(\gamma_0)^{-m}\zeta_{p^i} - 1) \neq 0.$$

If these assertions are valid, then the order of $M(m)^{\Gamma_n}$ is

$$\prod_{i=0}^{n} |f(\kappa(\gamma_0)^{-m}\zeta_{p^i} - 1)|_{v_i}^{-1}.$$

The order of the p-primary part of $K_{2i}(O_{F_n})$

I_n (resp. I): the free abelian group generated by the primes of F_n (resp. F_∞) which do not lie above p.

P_n (resp. P): the subgroup of principal p-ideals in I_n (resp. I).

$C_n = I_n/P_n$ (resp. $C = I/P$).

$\mathfrak{C}_n$ (resp. $\mathfrak{C}$): the p-primary component of C_n (resp. C).

O_F: the ring of integers in F.

$\mathfrak{O}_0 = O_F[\frac{1}{p}]$ and $\mathfrak{O}_n$ (resp. $\mathfrak{O}$) is the algebraic closure of $\mathfrak{O}_0$ in F_n (resp. F_∞).

$\mathcal{U}_n$ (resp. $\mathcal{U}$): the group of all p-units in F_n (resp. F_∞), i.e., the multiplicative group of the ring $\mathfrak{O}_n$ (resp. $\mathfrak{O}$).

Then we have

$$I = \varinjlim I_n, \;\; C = \varinjlim C_n, \;\; \mathfrak{C} = \varinjlim \mathfrak{C}_n, \;\; \mathcal{U} = \varinjlim \mathcal{U}_n.$$

There is a well defined surjective homomorphism

$$\psi : (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} F_\infty^\times \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} I.$$

We define $\mathfrak{M}$ to be its kernel.

Thus we have the exact sequence

$$0 \longrightarrow \mathfrak{M} \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} F_\infty^\times \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} I \longrightarrow 0.$$

Lemma 4.1. (Soule) For any integer $i \geq 2$, one has

$$\mathfrak{M}(i-1)_{\Gamma_n} = 0;$$

$$\mathfrak{M}(i-1)^{\Gamma_n} = H^1(\mathfrak{O}, W^{(i)})^{\Gamma_n}$$

$$= H^1(\mathfrak{O}_n, W^{(i)}) = (\mathbb{Q}_p/\mathbb{Z}_p)^{p^n d/2} \oplus G_{n,i},$$

where $G_{n,i}$ is a finite group.

Lemma 4.2. For any integers $n \geq 0$ and $i \geq 1$, we have

$$K_{2i}(O_{F_n})\{p\} \cong G_{n,i+1}.$$

This follows from

(1)

$$\mathfrak{M}(i-1)^{\Gamma_n} = (\mathbb{Q}_p/\mathbb{Z}_p)^{p^n d/2} \oplus G_{n,i} \quad \text{(Soule)}.$$

(2) Let O_S be the ring of S-integers in a number field F with some set S of finite places of F. If p is a prime, then

$$K_{2i}(O_S)\{p\} \cong H^2(O_S[\frac{1}{p}], \mathbb{Z}_p(i+1))$$

(Voevodsky, Rost, Suslin,..., See for example, C. Weibel's paper in Handkook of K-Theory, editors: E.M.Friedlander and D.R.Grayson, Springer 2005.)

(3) For all integers $n \geq 0$ and $i \geq 2$,

$$H^2(\mathfrak{O}_n, \mathbb{Z}_p(i)) \cong H^1(\mathfrak{O}_n, W^{(i)})/H^1(\mathfrak{O}_n, W^{(i)})_{\text{div}}.$$

(4)

$$H^2(\mathfrak{O}_n, \mathbb{Z}_p(i+1)) \cong K_{2i}(\mathfrak{O}_n)\{p\}$$

and

$$K_{2i}(O_{F_n})\{p\} \cong K_{2i}(\mathfrak{O}_n)\{p\}.$$

Let $f(T)$ be the characteristic polynomial of the Λ-module $\mathrm{Gal}(M/F_\infty)^\bullet$.

Theorem 4.3. For any $n \geq 0$ and $i \geq 1$, we have

$$\sharp K_{2i}(O_{F_n})\{p\} = \sharp H(i)^{\Gamma_n} \cdot \prod_{j=0}^{n} |f(\kappa(\gamma_0)^{-i}\zeta_{p^j} - 1)|_{v_j}^{-1},$$

where

$$H = \frac{\Lambda^{d/2}}{\mathrm{Gal}(M/F_\infty)^\bullet / t(\mathrm{Gal}(M/F_\infty)^\bullet)}$$

is a finite Λ-module.

Corollary 4.4. If $S(F_\infty/F) = 1$, i.e., F_∞ has only one prime divisor which is ramified for extension F_∞/F. Then for all integers $n \geq 0$ and $i \geq 1$, we have

$$\sharp K_{2i}(O_{F_n})\{p\} = \sharp H(i)^{\Gamma_n} \cdot \prod_{j=0}^{n} |h(\kappa(\gamma_0)^{-i}\zeta_{p^j} - 1)|_{v_j}^{-1},$$

where $h(T)$ is the characteristic polynomial of the Pontryagin dual of $\mathfrak{C}$.

Note that H finite implies, for sufficiently large n, $H(i)^{\Gamma_n} = H(i)$. So we have the following.

Corollary 4.5. Let $i \geq 1$. Then for sufficiently large n, we have

$$\sharp K_{2i}(O_{F_n})\{p\} = \sharp H \cdot \prod_{j=0}^{n} |f(\kappa(\gamma_0)^{-i}\zeta_{p^j} - 1)|_{v_j}^{-1}.$$

Corollary 4.6. The finite group H is trivial if and only if there exists integer $i \geq 1$ such that

$$\sharp K_{2i}(O_F)\{p\} = |f(\kappa(\gamma_0)^{-i} - 1)|_p^{-1}.$$

Theorem 4.7. (1) For any $i \geq 1$, if $K_{2i}(O_F)\{p\} = 0$, then $K_{2i}(O_{F_n})\{p\} = 0$, for all $n \geq 0$.

(2) For any $i \geq 1$, there exist integers n_i and ν_i such that, for all $n \geq n_i$,

$$e(i)_n = \lambda n + \mu p^n + \nu_i,$$

where $p^{e(i)_n} = \sharp K_{2i}(O_{F_n})\{p\}$, λ and μ are the classical Iwasawa invariants of the Λ-module Gal(M/F_∞) independent of i and n, and ν_i is a constant independent of n.

K-groups and ideal class groups

In this section,

p : an odd prime number;

$F = \mathbb{Q}(\zeta_p)$ the p-th cyclotomic field;

$F_n = \mathbb{Q}(\zeta_{p^{n+1}})$;

$F_\infty = \bigcup_{n \geq 0} F_n$;

$F^+ = \mathbb{Q}(\zeta_p)^+$;

$\Delta = \mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$;

ω: the Teichmuller character;

$\widehat{\Delta} = \{\omega^i | 0 \leq i \leq p-2\}$;

$\varepsilon_i = \frac{1}{p-1}\sum_{a=1}^{p-1} \omega^i(a)\sigma_a^{-1}, \quad 0 \leq i \leq p-2$;

$\varepsilon_- = \frac{1-\sigma_{-1}}{2} = \sum_{i \text{ odd}} \varepsilon_i$;

$\varepsilon_+ = \frac{1+\sigma_{-1}}{2} = \sum_{i \text{ even}} \varepsilon_i$;

For an $\mathbb{Z}_p[\Delta]$-module A,

$A^{(i)} = \varepsilon_i A$;

$A^- = \varepsilon_- A$;

$A^+ = \varepsilon_+ A$.

Recall that M is the maximal abelian p-extension of F_∞ unramified outsider p. Let L denote the maximal unramified abelian p-extension over F_∞ in M. Let N' be the field generated over F_∞ by the p^a-th roots of all elements ε in $\mathcal{U}$ for all integers $a \geq 0$.

K. Iwasawa, On $\mathbb{Z}_l$-extensions of algebraic number fields, Ann. Math. 98(1973), 246-326, shows the following:

(1) Gal$(M/N')^\bullet$ is isomorphic to the Pontryagin dual of $\mathfrak{C}$ and it is a Noetherian torsion Λ-module with no non-trivial finite Λ-submodule.

(2) Gal$(N'/F_\infty)^\bullet$ is isomorphic to the Pontryagin dual of $\mathcal{E}$, which is a torsion free $\mathbb{Z}_p$-module and is contained as a Λ-submodule of finite index in an elementary Λ-module of the form

$$\Lambda^{d/2} \oplus M$$

where $M = \oplus_{j=1}^{t} \Lambda/(g_j(T))$.

(3) Then the Galois group $\mathrm{Gal}(M/F_\infty)$ is a Noetherian Λ-module and has no non-trivial finite Λ-submodule. We have

$$0 \longrightarrow \mathrm{Gal}(M/F_\infty)^\bullet/t(\mathrm{Gal}(M/F_\infty)^\bullet) \longrightarrow \Lambda^{d/2} \longrightarrow H \longrightarrow 0.$$

Now assume that $F = \mathbb{Q}(\zeta_p)$.

Let K_n be the maximal unramified abelian p-extension over F_n and L_n be the maximal ablian extension over F_n in M. Write

$$\omega_n = \omega_n(T) = (1+T)^{p^n} - 1.$$

Then we have the following:

(i) $S(F_\infty/F) = 1$.

(ii) C_n is also the ideal class group of F_n. And $\mathfrak{C}_n$ is also the p-primary subgroup of the ideal class group of F_n.

(iii)

$$L_n = F_\infty K_n,$$

$$\omega_n \mathsf{Gal}(L/F_\infty) = \mathsf{Gal}(L/L_n)$$

$$(\mathsf{Gal}(L/F_\infty))_{\Gamma_n} = \mathsf{Gal}(L/F_\infty)/\omega_n \mathsf{Gal}(L/F_\infty)$$

$$\cong \mathsf{Gal}(L_n/F_\infty) \cong \mathsf{Gal}(K_n/F_n) \cong \mathfrak{C}_n.$$

(iv)

$$\mathsf{Gal}(M/N')^\bullet \cong \mathsf{Hom}(\mathfrak{C}, \mathbb{Q}_p/\mathbb{Z}_p) \cong \alpha(\mathsf{Gal}(L/F_\infty)) \sim \mathsf{Gal}(L/F_\infty),$$

where $\alpha(\mathsf{Gal}(L/F_\infty))$ is the adjoint of $\mathsf{Gal}(L/F_\infty)$ and $\sim$ means pseudo-isomorphism.

(v) Let Y be the Pontryagin dual of $\mathcal{E}$. Then

$$Y \cong \mathrm{Gal}(N'/F_\infty)^\bullet$$

and there is an exact sequence:

$$0 \longrightarrow Y \longrightarrow \Lambda^{\frac{p-1}{2}} \longrightarrow H \longrightarrow 0,$$

where

$$H = \frac{\Lambda^{\frac{p-1}{2}}}{\mathrm{Gal}(M/F_\infty)^\bullet / t(\mathrm{Gal}(M/F_\infty)^\bullet)}$$

is finite.

(vi) Let $f(T)$ be the characteristic polynomial of $\mathrm{Gal}(L/F_\infty)$. Then $f(T)$ is also the characteristic polynomial of the Λ-module $\mathrm{Gal}(M/N')^\bullet$ and $\mathrm{Gal}(M/F_\infty)^\bullet$.

(vii) Let $X = \mathrm{Gal}(L/F_\infty)$. Then X^- has no non-trivial finite Λ-submodule and there are exact sequences:

$$0 \longrightarrow A_i \longrightarrow X^{(i)} \longrightarrow \oplus_{j=1}^{t_i} \Lambda/(f_{i,j}(T)) \longrightarrow B_i \longrightarrow 0$$

where A_i and B_i are finite Λ-submodules and $A_i = 0$ if i is odd.

Now set

$$A = \bigoplus_{i \text{ is even}} A_i$$

$$B^+ = \bigoplus_{i \text{ is even}} B_i,\ B^- = \bigoplus_{i \text{ is odd}} B_i,$$

$$B = B^+ \oplus B^-,$$

$$f_i(T) = \prod_{j=1}^{t_i} f_{i,j}(T),$$

$$f^{+} = \prod_{i \text{ is even}} f_i(T),$$

$$f^{-} = \prod_{i \text{ is odd}} f_i(T),$$

$$\lambda = \lambda(X) = \deg f(T)$$

$$\lambda_i = \lambda(X^{(i)}) = \deg f_i(T),$$

$$\lambda^{+} = \lambda(X^{+}) = \deg f^{+}(T),$$

$$\lambda^{-} = \lambda(X^{-}) = \deg f^{-}(T),$$

Then

$$f_i(T),\ f^{+}(T),\ f^{-}(T)$$

are the characteristic polynomials of the Λ-modules $X^{(i)}$, X^{+} and X^{-}, respectively. So

$$f(T) = \prod_{2 \le i \le p-2} f_i(T) = f^{+}(T) f^{-}(T)$$

and there are exact sequences:

$$0 \longrightarrow A \longrightarrow X^{+} \longrightarrow \bigoplus_{i \text{ is even}} \bigoplus_{j=1}^{t_i} \Lambda/(f_{i,j}(T)) \longrightarrow B^{+} \longrightarrow 0,$$

$$0 \longrightarrow X^{-} \longrightarrow \bigoplus_{i \text{ is odd}} \bigoplus_{j=1}^{t_i} \Lambda/(f_{i,j}(T)) \longrightarrow B^{-} \longrightarrow 0,$$

$$0 \longrightarrow A \longrightarrow X \longrightarrow \bigoplus_{2 \le i \le p-2} \bigoplus_{j=1}^{t_i} \Lambda/(f_{i,j}(T)) \longrightarrow B \longrightarrow 0.$$

(vi) If Vandiver's conjecture holds for p, then

$$X^{(i)} = \varepsilon_i X \cong \Lambda/(f(T, \omega^{1-i}))$$

for $i = 3, 5, \cdots, p-2$, where

$$f((1+p)^s - 1, \omega^{1-i}) = L_p(s, \omega^{1-i}).$$

Factor $f(T, \omega^{1-i}) = p^{\mu_i} g_i(T) U_i(T)$ with g_i distinguished if $g_i \neq 1$ and $U_i \in \Lambda^\times$. We know that $\mu_i = 0$. Therefore

$$X^{(i)} \cong \Lambda/(g_i(T))$$

and

$$X = X^- \cong \oplus_{\{i \neq 1 \ \ odd\}} \Lambda/(g_i(T)).$$

(viii) The finite Λ-module H is trivial if and only if $H^1(\Gamma, \mathcal{U}) = 1$ if and only if Vandiver's conjecture holds for p which implies A and B are trivial and

$$\mathrm{Gal}(M/N')^{\bullet} \cong \mathrm{Hom}(\mathfrak{C}, \mathbb{Q}_p/\mathbb{Z}_p) \cong \alpha(X) \cong X \cong \oplus_{\{i\neq 1 \ \ odd\}}\Lambda/(g_i(T)).$$

We can prove that the following exact sequence of Λ-modules :

$$1 \longrightarrow G(M/N')^{\bullet} \longrightarrow G(M/F_{\infty})^{\bullet} \longrightarrow G(N'/F_{\infty})^{\bullet} \longrightarrow 1$$

is split. So, we have

Lemma 5.1. The following sequence of Λ-modules is split:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{C} \longrightarrow 0,$$

where $\mathcal{E} = \mathcal{U} \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

Lemma 5.2. Let $r \geq 1$ and $n \geq 0$. Then there are the following isomorphism

$$K_{2r}(O_{F_n})\{p\}^{(i)} \cong \mathfrak{C}(r)^{\Gamma_n(i)}, \ \ i = 3, 5, \cdots, p-2,$$

and exact sequence of abelian groups

$$0 \longrightarrow H(r)^{\Gamma_n} \longrightarrow K_{2r}(O_{F_n})\{p\}^{+} \longrightarrow \mathfrak{C}(r)^{\Gamma_n +} \longrightarrow 0.$$

Theorem 5.3. (1) The odd prime number p is regular if and only if there exist integers $i \geq 1$ and $n \geq 0$ such that $K_{2i}(O_{F_n})\{p\}$ is trivial, if and only if for all integers $i \geq 1$ and $n \geq 0$ such that $K_{2i}(O_{F_n})\{p\}$ is trivial.

(2) ${\mathfrak{C}_0}^{(i)} = 0$ if and only if

${\mathfrak{C}_n}^{(i)} = 0$ for some $n \geq 0$ if and only if

${\mathfrak{C}_n}^{(i)} = 0$ for all $n \geq 0$,

in the case, $\lambda_i = 0$.

Further more, if i is odd, then $\lambda_i = 0$ implies ${\mathfrak{C}_0}^{(i)} = 0$.

(3) Let $i = 3, 5, \cdots, p-2$. Then

$K_{2r}(O_{F_0})\{p\}^{(i)} = 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^{(i)} = 0$ for some $n \geq 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^{(i)} = 0$ for all $n \geq 0$ if and only if

$\lambda_i = 0$.

$K_{2r}(O_{F_0})\{p\}^{+} = 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^{+} = 0$ for some $n \geq 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^{+} = 0$ for all $n \geq 0$, and in this case, $\lambda^{+} = 0$.

Lemma 5.4. (a) Let $M = \Lambda/(g(T))$ with $g(T)$ a distinguished polynomial and $g(T)$ and $\omega_n(T)$ relatively prime. For all integers $i \geq 1$ and $n \geq 0$, we have

$$p\text{-rk}(M(i)_{\Gamma_n}) = p\text{-rk}(M_{\Gamma_n})$$
$$= \min\{p^n, \deg(g(T))\}.$$

Moreover, let n_0 be the smallest integer such that $p^{n_0} \geq \deg(g(T))$. Then there exist integers $n_1, n_2, \cdots, n_d$, where $d = \deg(g(T))$, such that for all $n \geq n_0 + 1$, we have

$$\frac{\Lambda}{(w_n, g)} \cong \bigoplus_{i=1}^{d} p^{-n-n_i}\mathbb{Z}_p/\mathbb{Z}_p.$$

(b) Let X be a Noetherian torsion Λ-module such that $\mu(X) = 0$ and X_{Γ_n} is finite for all $n \geq 0$. Then

$$p\text{-rk}(X_{\Gamma_n}) \geq \lambda(X),$$

$$p\text{-rk}(X(i)_{\Gamma_n}) \geq \lambda(X), \quad n \gg 0.$$

Further more, if X has no finite Λ-submodule, then for any integers i, there exist integers $n_0,\ n_1,\ n_2,\ \cdots, n_{\lambda(X)},\ \nu_1,\ \nu_2, \cdots, \nu_{\lambda(X)}$, such that for all $n \geq n_0$, we have

$$p\text{-rk}(X_{\Gamma_n}) = p\text{-rk}(X(i)_{\Gamma_n}) = \lambda(X),$$

and

$$X_{\Gamma_n} \cong \bigoplus_{j=1}^{\lambda(X)} p^{-n-n_j}\mathbb{Z}_p/\mathbb{Z}_p,$$

$$X(i)_{\Gamma_n} \cong \bigoplus_{j=1}^{\lambda(X)} p^{-n-\nu_j}\mathbb{Z}_p/\mathbb{Z}_p.$$

Corollary 5.5. (1) Let $i = 3, 5, \cdots, p-2$ be odd. Then there exist integers n_0, $n_{i1}, \cdots, n_{i\lambda_i}$ such that for all $n \geq n_0$ we have

$$p\text{-rk}(\mathfrak{C}_n{}^{(i)}) = \lambda_i,\ \mathfrak{C}_n{}^{(i)} \cong \bigoplus_{j=1}^{\lambda_i} p^{-n-n_{ij}}\mathbb{Z}_p/\mathbb{Z}_p,$$

hence

$$p\text{-rk}(\mathfrak{C}_n{}^{-}) = \lambda^{-},$$

$$\mathfrak{C}_n{}^{-} \cong \bigoplus_{3 \leq i \text{ is odd}} \bigoplus_{j=1}^{\lambda_i} p^{-n-n_{ij}}\mathbb{Z}_p/\mathbb{Z}_p,\ \ n \gg 0.$$

(2) Let $i = 2, 4, \cdots, p-3$ be even. Then there exists integer n_0 such that for all $n \geq n_0$ we have

$$p\text{-rk}(\mathfrak{C}_n{}^{(i)}) \geq \lambda_i,\ \ p\text{-rk}(\mathfrak{C}_n{}^{+}) \geq \lambda^{+}.$$

(3) Let $r \geq 1$ and $i = 3, 5, \cdots, p{-}2$. There exist integers $n_0, n_1, \cdots, n_\lambda$ such that for all $n \geq n_0$, we have the following isomorphisms of abelian groups:

$$K_{2r}(O_{F_n})\{p\}^{(i)} \cong \bigoplus_{j=1}^{\lambda_i} p^{-n-n_j}\mathbb{Z}_p/\mathbb{Z}_p,$$

and

$$K_{2r}(O_{F_n})\{p\}^{+} \cong H \oplus \bigoplus_{j=1}^{\lambda^+} p^{-n-m_j}\mathbb{Z}_p/\mathbb{Z}_p, \ \ n \gg 0,$$

for some integers $m_1, \cdots, m_{\lambda^+}$ independent of n.

(4)

$$p\text{-rk}(X_{\Gamma_n}) = p\text{-rk}(H) + p\text{-rk}(\alpha(X)_{\Gamma_n}),$$

hence

$$p\text{-rk}({\mathfrak{C}_n}^+) = p\text{-rk}(H) + \lambda^+.$$

Corollary 5.6. Let $i = 3, 5, \cdots, p-2$ be odd. Then $\lambda_i = 1$ if and only if ${\mathfrak{C}_n}^{(i)}$ is a cyclic group if and only if $K_{2r}(O_{F_n})\{p\}^{(i)}$ is a cyclic group, and in this case,

$$ {\mathfrak{C}_n}^{(i)} \cong \frac{\mathbb{Z}_p}{(w_n(a_i))}, $$

$$ K_{2r}(O_{F_n})\{p\}^{(i)} \cong \frac{\mathbb{Z}_p}{(w_n((1+p)^r(1+a_i)-1))}, $$

where a_i is the root of the characteristic polynomial of $X^{(i)}$, i.e.,

$$ f(T, w^{1-i}) = (T - a_i)U_i(T), \; U_i(T) \in \Lambda^*, $$

where $f((1+p)^s - 1, \omega^{1-i}) = L_p(s, \omega^{1-i})$.

Corollary 5.7. Let λ denote the Iwasawa λ-invariant of the Λ-module Gal(L/F_∞). Then the following statements are equivalent:

(a) Vandiver conjecture holds for p;

(b) The finite Λ-module H in is trivial;

(c) The finite Λ-modules A and B are trivial;

(d) X is an elementary Λ-module;

(e) $\alpha(X) \cong X$;

(f) p-rk$(X_{\Gamma_n}) = p$-rk$(\alpha(X)_{\Gamma_n})$ for some $n \gg 0$;

(g) p-rk$(X_{\Gamma_n}) = \lambda$ for some $n \gg 0$;

(g') p-rk$(X^+{}_{\Gamma_n}) = \lambda^+$ for some $n \gg 0$;

(h) p-rk(Cl$(O_{F_n})) = \lambda$ for some $n \gg 0$;

(h') p-rk(Cl$(O_{F_n})^+) = \lambda^+$ for some $n \gg 0$;

(j) for any $i \geq 1$, p-rk$(K_{2i}(O_{F_n})) = \lambda$ for some $n \gg 0$;

(j') for any $i \geq 1$, p-rk$(K_{2i}(O_{F_n^+})) = \lambda^+$ for some $n \gg 0$.

If these statements hold, then $\lambda^+ = 0$, i.e., $\lambda = \lambda^-$.

Remarks. (1) (Kurihara) ${\mathfrak{C}_0}^{(p-3)}$ always vanishes.

(2) (Soule) ${\mathfrak{C}_0}^{(p-n)}$ is trivial if $logp > n^{224n^4}$ odd.

Theorem 5.8. Let p be an odd prime and assume Vandiver conjecture holds for p. Let $i_1, \cdots, i_s$ be the even indices i such that $2 \leq i \leq p-3$ and $p|B_i$. If

$$B_{1,\omega^{i-1}} \not\equiv 0 \mod p^2$$

and

$$\frac{B_i}{i} \not\equiv \frac{B_{i+p-1}}{i+p-1} \mod p^2 \quad \text{for all } i \in \{i_1, \cdots, i_s\},$$

then

(1)

$$X \cong \oplus_{i \in \{i_1, \cdots, i_s\}} \Lambda/(T - \alpha_i),$$

$$\mathfrak{C}_n \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^s, \quad for \ all \ n \geq 0,$$

where $\alpha_i \in p\mathbb{Z}_p$ and $v_p(\alpha_i) = 1$ for all $i \in \{i_1, \cdots, i_s\}$.

(2) For all integers $m \geq 1$ and $n \geq 0$, we have

$$K_{2m}(O_{F_n})\{p\} \cong \oplus_{i \in \{i_1, \cdots, i_s\}} \mathbb{Z}/p^{n+1+c_i}\mathbb{Z}$$

where $c_i = \nu_p((1+p)^m(1+\alpha_i) - 1) - 1$ for all $i \in \{i_1, \cdots, i_s\}$. In particular, if $m + \frac{\alpha_i}{p} \not\equiv 0 \mod p$ for all $i \in \{i_1, \cdots, i_s\}$, then

$$K_{2m}(O_{F_n})\{p\} \cong \mathrm{Cl}(O_{F_n})\{p\} \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^s.$$

Here B_i and $B_{1,\omega^{i-1}}$ are respectively the ordinary Bernoulli numbers and the generalized Bernoulli numbers.

Corollary 5.9. Let p be an odd prime and assume Vandiver conjecture holds for p. Let $i_1, \cdots, i_s$ be the even indices which satisfy conditions of Theorem 5.8. Then for all integers $n \geq 0$ and $m \geq 1$ such that $m \not\equiv -\frac{B_{1,\omega^{i-1}}}{B_{2,\omega^{i-2}}/2 - B_{1,\omega^{i-1}}} \pmod{p}$ for all $i \in \{i_1, \cdots, i_s\}$, we have

$$K_{2m}(O_{F_n})\{p\} \cong \mathrm{Cl}(O_{F_n})\{p\} \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^s.$$