Calculation of l-adic Local Fourier Transformations

Lei Fu

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A $\overline{\mathbf{Q}}_l$-representation of the galois group of the function field of X unramified on X gives rise to a $\overline{\mathbf{Q}}_l$-sheaf on X. A complex of such galois representations gives an object in $D(X, \overline{\mathbf{Q}}_l)$.

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Analogue: $\hat{f}(t') = \int_{-\infty}^{\infty} f(t) e^{itt'} dt$.

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Analogue: $\int_{-\infty}^{\infty} g(x)e^{if(x)t'}dx$, where $g(x)$ is supported in $[a, b]$.

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= \quad & O(\frac{1}{t'^n}) \text{ for all } n
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We would like to have an l-adic analogue of the stationary phase principle.

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A perverse sheaf K on X is called **unramified** at a closed point x of X if there exists an étale neighborhood U of x and a lisse $\overline{\mathbf{Q}}_l$-sheaf $\mathcal{G}$ on U such that $K|_U \cong \mathcal{G}[1]$.

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$\mathcal{F}(K)|_{\eta_{\infty'}}$ only depends on $K|_{\eta_x}$ ($x \in \mathbf{A}^1 - U$) and $K|_{\eta_\infty}$.

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Consider a formal Laurent series

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Laumon-Malgrange Conjecture. If $p \gg r, s$, then we have

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Study of $[r+s]^*\mathcal{F}(j_![r]_*\alpha^*\mathcal{L}_\psi)$

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One can check directly

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& \mathrm{Tr}\Big(F_{t'}, \Big([r+s]^*\mathcal{F}(j_![r]_*\alpha^*\mathcal{L}_\psi)\Big)_{\bar{t}'}\Big) \\
= & \sum_{x\in(\mathbf{A}^1-\{0\})(\mathbf{F}_{q^k})} \psi_k(x^r t'^{r+s} + 1/x^s) \\
= & \sum_{x\in(\mathbf{A}^1-\{0\})(\mathbf{F}_{q^k})} \psi_k(t'^s((xt')^r + 1/(xt')^s)) \\
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that is,

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The proof of the Laumon-Malgrange conjecture for general α involves more complicated changes of variables, The proof of the Laumon-Malgrange conjecture for general α involves more complicated changes of variables, and a subtle use of a fact stronger than the Stationary Phase Principle.