Kashaev's volume conjecture

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(Complexified) Kashaev's volume conjecture

Conjecture

$$vol(L) = 2\pi \lim_{N\to\infty} \frac{\log |\langle L \rangle_N|}{N},$$

where L is a hyperbolic link, vol(L) is the hyperbolic volume, $< L >_N$ *is the Kashaev invariant.*

Conjecture

$$i(vol(L) + i\, cs(L)) \equiv 2\pi \lim_{N\to\infty} \frac{\log\langle L \rangle_N}{N} \pmod{\pi^2},$$

where cs(L) is the Chern-Simons invariant.

Hyperbolic space

Definition (Upper half space model of $\mathbb{H}^3$)

$$\mathbb{H}^3 = \{(z,t) | z \in \mathbb{C}, t > 0\}$$

with the metric

$$ds^2 = \frac{dz^2 + dt^2}{t^2}$$

is called the *hyperbolic space*. $\mathbb{H}^3$ is a Riemannian 3-manifold with the constant sectional curvature -1.

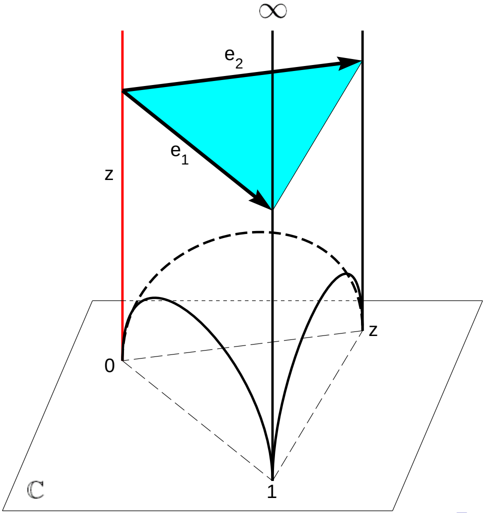
We consider $\partial\mathbb{H}^3 = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition

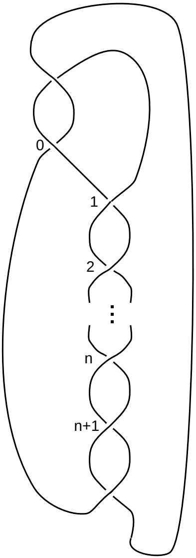
A knot K is called *hyperbolic* if the complement $S^3 - K$ admits a complete hyperbolic structure.

Lemma

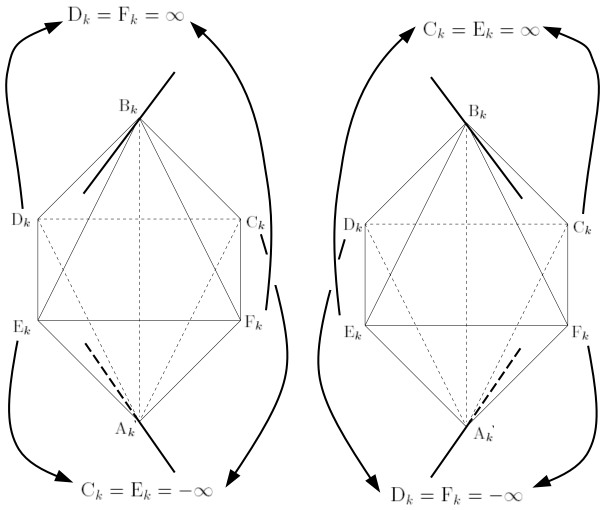
An ideal tetrahedron in $\mathbb{H}^3$ can be parametrized with a complex number $z \in \mathbb{C}$.



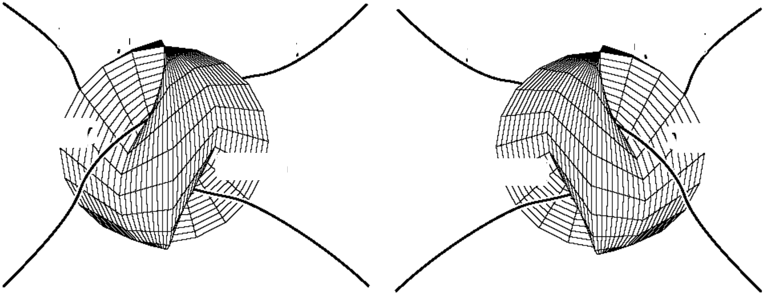
Definition of the Twist Knot T_n



Ideal Triangulation

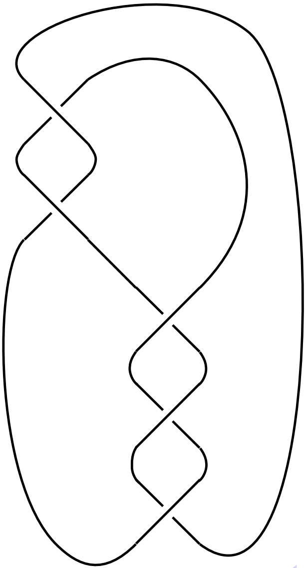


Ideal Triangulation

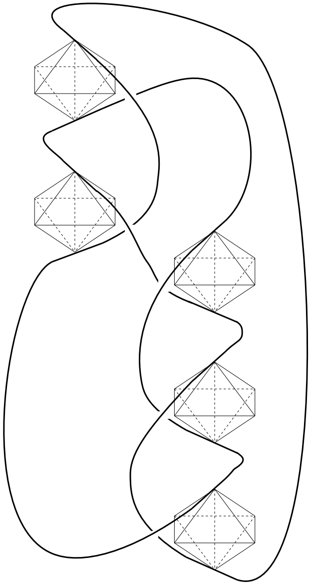


This picture comes from the paper of H. Murakami.

Example of 5_2 Knot



Gluing Pattern of the Ideal Triangulation



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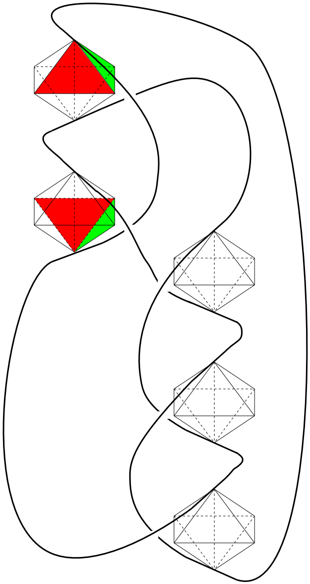

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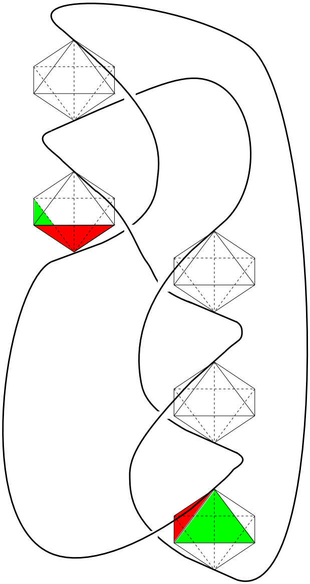

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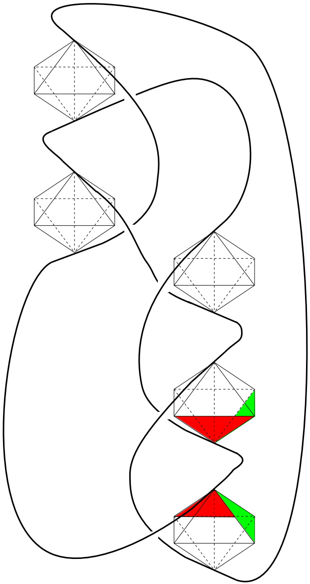
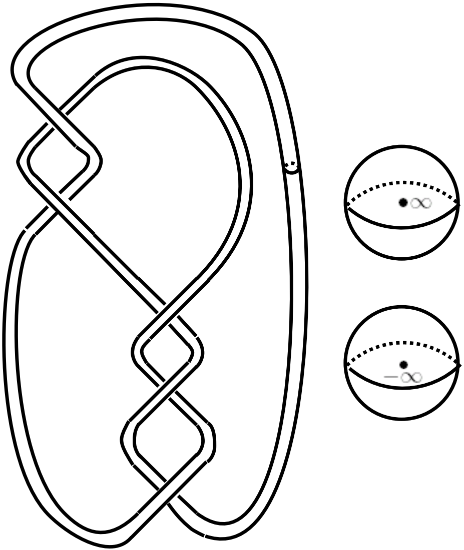
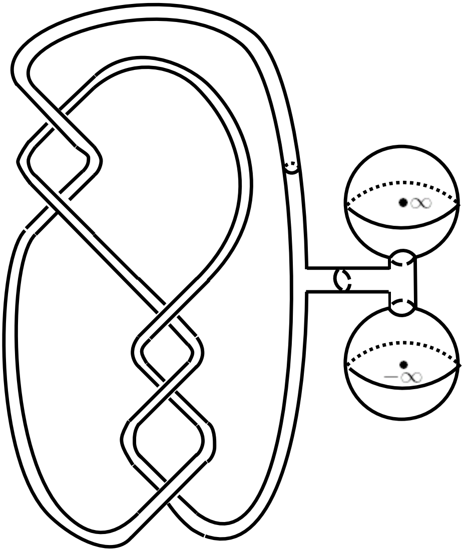


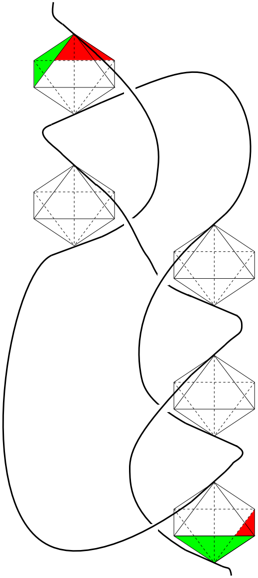
Result of the Gluing



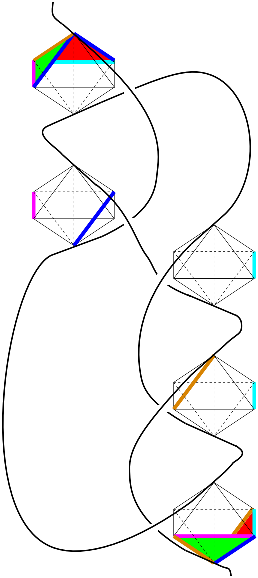
Ideal Triangulation of the 5_2 Knot Complement



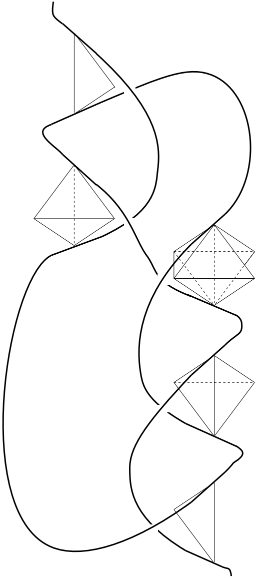
Collapsing Tetrahedra



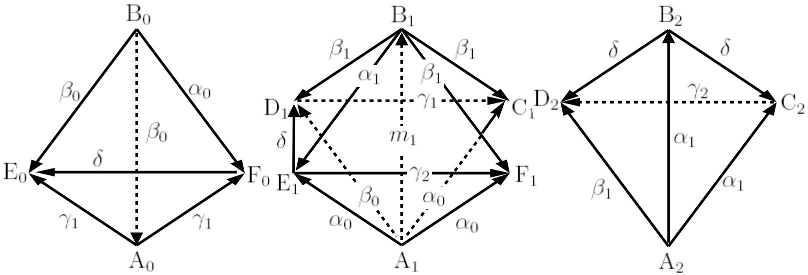
Collapsing Tetrahedra



Topological Ideal Triangulation of the 5_2 Knot Complement

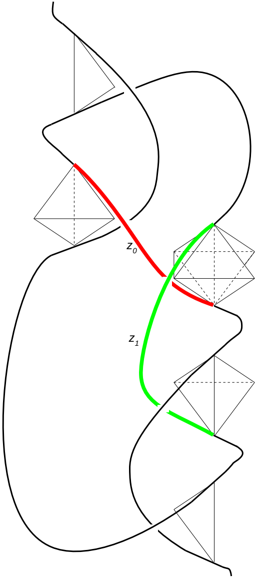


Topological Ideal Triangulation of the 5_2 Knot Complement

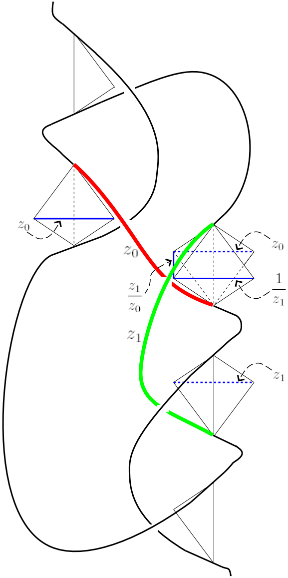


(Note that $\beta_0 = \beta_1$ and $\alpha_0 = \delta$)

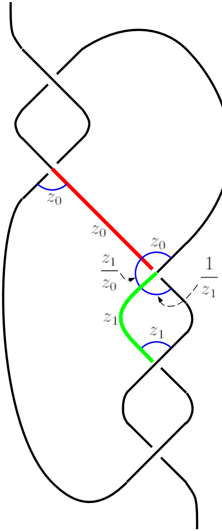
Parametrizing Tetrahedra



Parametrizing Tetrahedra



Parametrization of Tetrahedra



Hyperbolicity Equation

Using this triangulation, the hyperbolicity equations of 5_2 knot is as follows.

$$\begin{cases} 1 - \frac{z_0}{z_1} = (1 - z_0)(1 - \frac{1}{z_0}), \\ 1 - \frac{1}{z_1} = (1 - z_1)(1 - \frac{z_0}{z_1}). \end{cases}$$

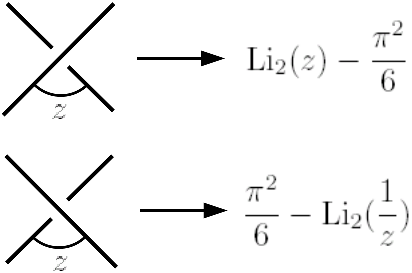
There is unique solution (z_0, z_1) which gives the hyperbolic structure to the 5_2 knot complement. We call the unique solution the *geometric solution*. Let r_k be the even integers satisfying

$$r_k \pi i = \begin{cases} \log(1 - \frac{z_0}{z_1}) - \log(1 - z_0) - \log(1 - \frac{1}{z_0}) & \text{for } k = 0, \\ \log(1 - \frac{1}{z_1}) - \log(1 - z_1) - \log(1 - \frac{z_0}{z_1}) & \text{for } k = 1. \end{cases}$$

Using numerical calculation, we obtain

$$-(1-z_0)^3 = z_0 = 0.3376410214 + 0.5622795125i, \; z_1 = z_0 - 1, \; r_0 = r_1 = 0.$$

Definition of $V(z_0, z_1)$ and $V_0(z_0, z_1)$



Definition of $V(z_0, z_1)$ and $V_0(z_0, z_1)$

Let $V(z_0, z_1)$ be

$$\begin{aligned} V(z_0, z_1) &= \left\{\frac{\pi^2}{6} - \mathrm{Li}_2\left(\frac{1}{z_0}\right)\right\} \\ &+ \left\{\mathrm{Li}_2(z_0) - \frac{\pi^2}{6}\right\} + \left\{\frac{\pi^2}{6} - \mathrm{Li}_2\left(\frac{z_0}{z_1}\right)\right\} + \left\{\mathrm{Li}_2\left(\frac{1}{z_1}\right) - \frac{\pi^2}{6}\right\} \\ &+ \left\{\mathrm{Li}_2(z_1) - \frac{\pi^2}{6}\right\} \end{aligned}$$

and $V_0(z_0, z_1)$ be

$$V_0(z_0, z_1) = V(z_0, z_1) - \sum_{k=0}^{1} r_k \pi i \log z_k.$$

Optimisitc limit of $\langle 5_2 \rangle_N$

By applying the formal approximation

$$\frac{1}{(q)_k} \sim \exp \frac{N}{2\pi i}\left(\mathrm{Li}_2(q^k) - \frac{\pi^2}{6}\right),\ \frac{1}{(\overline{q})_k} \sim \exp \frac{N}{2\pi i}\left(\frac{\pi^2}{6} - \mathrm{Li}_2(\overline{q}^k)\right)$$

to

$$\langle 5_2 \rangle_N = \pm \sum_{1 \le k_1 \le k_2 + 1 \le N} \frac{N^3}{(\overline{q})_{k_1-1}(q)_{k_1-1}(\overline{q})_{k_2-k_1+1}(q)_{N-k_2-1}(q)_{k_2}}.$$

and by letting $z_0 = q^{k_1},\ z_1 = q^{k_2}$, we can obtain $V(z_0, z_1)$ again as follows:

$$\begin{aligned}\frac{2\pi i \log\langle 5_2 \rangle}{N} &\sim \left\{\frac{\pi^2}{6} - \mathrm{Li}_2(\frac{1}{z_0})\right\} \\ &+ \left\{\mathrm{Li}_2(z_0) - \frac{\pi^2}{6}\right\} + \left\{\frac{\pi^2}{6} - \mathrm{Li}_2(\frac{z_0}{z_1})\right\} + \left\{\mathrm{Li}_2(\frac{1}{z_1}) - \frac{\pi^2}{6}\right\} \\ &+ \left\{\mathrm{Li}_2(z_1) - \frac{\pi^2}{6}\right\} = V(z_0, z_1).\end{aligned}$$

Optimisitc limit of $\langle 5_2 \rangle_N$

Note that

$$\begin{cases} z_0 \frac{\partial V(z_0,z_1)}{\partial z_0} = \log(1 - \frac{z_0}{z_1}) - \log(1 - z_0) - \log(1 - \frac{1}{z_0}) = r_0 \pi i, \\ z_1 \frac{\partial V(z_0,z_1)}{\partial z_1} = \log(1 - \frac{1}{z_1}) - \log(1 - z_1) - \log(1 - \frac{z_0}{z_1}) = r_1 \pi i. \end{cases}$$

We define the optimistic limit of $\langle 5_2 \rangle_N$ by

$$\begin{aligned} \operatorname*{o-lim}_{N \to \infty} \frac{2\pi i \log \langle 5_2 \rangle_N}{N} &:= V(z_0, z_1) - \sum_{k=0}^{1} \left(z_k \frac{\partial V(z_0, z_1)}{\partial z_k} \log z_k \right) \\ &= V(z_0, z_1) - \sum_{k=0}^{1} r_k \pi i \log z_k = V_0(z_0, z_1). \end{aligned}$$

Yokota Theory for the 5_2 Knot

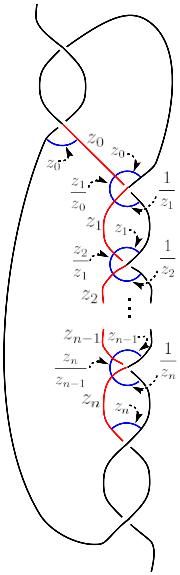
Theorem (Yokota)

Let $V(z_0, z_1)$ and $V_0(z_0, z_1)$ be the functions defined for the 5_2 knot. Then

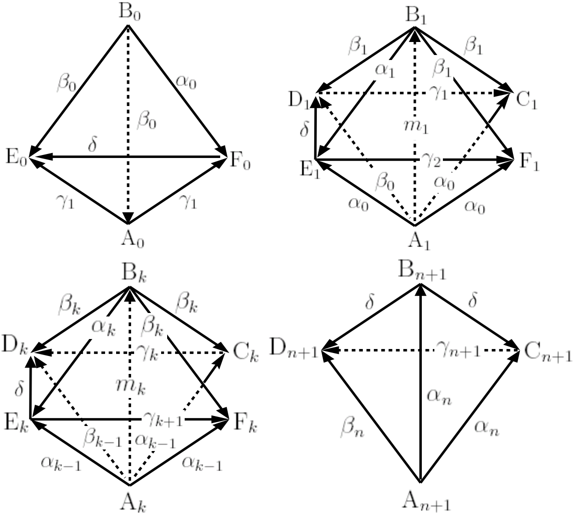
1. *$V(z_0, z_1)$ can be obtained from $\langle 5_2 \rangle_N$ by using formal substitution,*
2. $\left\{ \exp(z_0 \frac{\partial V(z_0,z_1)}{\partial z_0}) = 1, \exp(z_1 \frac{\partial V(z_0,z_1)}{\partial z_1}) = 1 \right\}$ *is the set of the hyperbolicity equations of the 5_2 knot,*
3. $\mathrm{Im}\, V_0(z_0, z_1) = \mathrm{vol}(5_2)$ *for the geometric solution (z_0, z_1).*

Now we will see the Yokota theory for the general twist knots.

Ideal Triangulation of the Twist Knot T_n



Ideal Triangulation of the Twist Knot T_n



$(k = 2, 3, \ldots, n$. Note that $\alpha_{n-1} = \delta$ and $\alpha_n = \gamma_1)$

Hyperbolicity Equation

Using this triangulation, the hyperbolicity equations of T_n is as follows.

$$\begin{aligned} 1-\frac{z_0}{z_1} &= (1-z_0)(1-\frac{1}{z_0}), \\ (1-\frac{z_k}{z_{k+1}})(1-\frac{1}{z_k}) &= (1-z_k)(1-\frac{z_{k-1}}{z_k}), \quad \text{for } k=1,2,\ldots,n-1, \\ 1-\frac{1}{z_n} &= (1-z_n)(1-\frac{z_{n-1}}{z_n}). \end{aligned}$$

Let r_k be the even integers satisfying

$$r_k\pi i = \begin{cases} \log(1-\frac{z_0}{z_1}) - \log(1-z_0) - \log(1-\frac{1}{z_0}) \qquad (k=0), \\ \log(1-\frac{z_k}{z_{k+1}}) + \log(1-\frac{1}{z_k}) - \log(1-z_k) - \log(1-\frac{z_{k-1}}{z_k}) \\ \qquad (k=1,2,\ldots,n-1), \\ \log(1-\frac{1}{z_n}) - \log(1-z_n) - \log(1-\frac{z_{n-1}}{z_n}) \qquad (k=n). \end{cases}$$

Definition of $V(z_0, z_1, \ldots, z_n)$ and $V_0(z_0, z_1, \ldots, z_n)$

Let $V(z_0, z_1, \ldots, z_n)$ be

$$V(z_0, z_1, \ldots, z_n) = \left(\frac{\pi^2}{6} - \mathrm{Li}_2(\frac{1}{z_0})\right)$$
$$+ \sum_{k=1}^{n} \left\{ \left(\mathrm{Li}_2(z_{k-1}) - \frac{\pi^2}{6}\right) + \left(\frac{\pi^2}{6} - \mathrm{Li}_2(\frac{z_{k-1}}{z_k})\right) + \left(\mathrm{Li}_2(\frac{1}{z_k}) - \frac{\pi^2}{6}\right) \right\}$$
$$+ \left(\mathrm{Li}_2(z_n) - \frac{\pi^2}{6}\right)$$

and $V_0(z_0, z_1, \ldots, z_n)$ be

$$V_0(z_0, z_1, \ldots, z_n) = V(z_0, z_1, \ldots, z_n) - \sum_{k=0}^{n} r_k \pi i \log z_k .$$

Yokota Theory for T_n

Theorem (Yokota)

Let $V(z_0, z_1, \ldots, z_n)$ and $V_0(z_0, z_1, \ldots, z_n)$ be the functions defined for the twist knot T_n. Then

1. *$V(z_0, z_1, \ldots, z_n)$ can be obtained from $\langle T_n \rangle_N$ by using formal substitution,*
2. *$\left\{ \exp\left(z_k \frac{\partial V(z_0, z_1, \ldots, z_n)}{\partial z_k}\right) = 1 \mid k = 0, 1, \ldots, n \right\}$ is the set of the hyperbolicity equations of T_n,*
3. *$\operatorname{Im} V_0(z_0, z_1, \ldots, z_n) = \operatorname{vol}(T_n)$ for the geometric solution $(z_0, z_1, \ldots, z_n)$.*

Recent developments

Theorem (Cho, J. Murakami and Yokota)

For a twist knot T_n and the geometric solution $(z_0, z_1, \ldots, z_n)$,

$$V_0(z_0, z_1, \ldots, z_n) \equiv i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \pmod{\pi^2}$$

where $\mathrm{cs}(T_n)$ is the Chern-Simons invariant of the T_n knot complement.

Theorem (Cho and J. Murakami)

For a twist knot T_n and the geometric solution $(z_0, z_1, \ldots, z_n)$,

$$\sum_{k=0}^{n} r_k \pi i \log z_k \equiv 0 \pmod{\pi^2}.$$

This implies

$$V(z_0, z_1, \ldots, z_n) \equiv i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \pmod{\pi^2}.$$

Recent developments

Lemma (H. Murakami and J. Murakami)

For a knot K,

$$\langle K\rangle_N = J_N\left(K;\exp(\frac{2\pi i}{N})\right)$$

where $J_N(K;u)$ is the N-th colored Jones polynomial of the knot K evaluated at $u\in\mathbb{C}$.

Ohnuki made 'the colored Jones polynomial version' of Yokota theory for 2-bridge links. Cho and J. Murakami reconstructed his theory for twist knots, and showed the relation between 'the volume and the Chern-Simons invariant of the knot complement' and 'the optimistic limit of the colored Jones polynomial'.