Finitely Dominated Spaces and Projective Modules

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Definition

Let $A \subset X$ be topological spaces.

(a) A continuous map $r : X \to A$ is called a *retraction* if $r(a) = a$ for all $a \in A$, i.e., $r \circ i = \mathrm{id}_A$, where $i : A \hookrightarrow X$. In this case, A is called a retract of X.

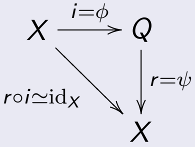
(b) A : *neighborhood retract* of X if $\exists$ open U such that $A \subseteq U \subseteq X$ and A is a retract of U.

(c) X : *absolute (neighborhood) retract* $(ANR)AR$ if for every normal space Y that embeds X as a closed subset, X is a (neighborhood) retract of Y.

Fact

(a) *A compact ANR (e.g. compact topological n-manifold) has the homotopy type of a CW-complex.*

(b) *If X is a compact ANR and $\phi : X \to P$ a homotopy equivalence with P a CW-complex then, as $\phi(X)$ will be contained in a finite subcomplex Q of P, the restriction to Q of a homotopy inverse to ϕ shows that X is a retract up to homotopy of the finite complex Q.*



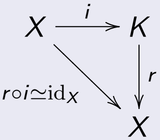
Conjecture (Borsuk's conjecture, 1954, International congress in Amsterdam)

A compact (metric) ANR is homotopy equivalent to a finite CW-complex.

Yes, i.e, Borsuk's conjecture is a theorem by West(1979) and Chapman(1980) independently via different approaches.
Raniki-Yamasaki proved also it by Controlled K-theory (1995).
(In fact, there were partial answers by Borsuk, Eckmann-Hilton, Kirby-Siebenmann).

Definition (Natural generalization of the above)

A topological space X is called *finitely dominated* if there exists a finite complex K such that X is a retract of K in homotopy category.



Fact

(a) *A compact ANR is finitely dominated.*

(b) *A finitely dominated space X is homotopy equivalent to a CW-complex.*

Question (J. H. C. Whitehead and J. Milnor)

Is a finitely dominated space X actually homotopy equivalent to a finite CW-complex?

Theorem (Mather, 1965)

X : *finitely dominated if and only if* $X \times S^1 \simeq$ *a finite CW-complex.*

Definition

1. An *infinite cyclic cover* of a path-connected space X is a covering space with fiber $\mathbb{Z}$.
2. Let $p : Y \to X$ be a covering space, and let $f : A \to X$ be a continuous map. The *pullback cover* of Y by f is defined to be the space $f^*Y = \{(a, y) = A \times Y \mid f(a) = p(y)\}$.

$$\begin{array}{ccc} f^*Y & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & X \end{array}$$

where q_1 and q_2 are canonical projection maps.

Proposition

The fibre of a pullback cover is homeomorphic to the fibre of the original cover.

Example

One importance case of infinite cyclic covers are those which are obtained as pullback covers of $\mathbb{R}$ by maps to S^1, i.e, spaces $f^*\mathbb{R}$, where $f : X \to S^1$:

$$\begin{array}{ccc} f^*\mathbb{R} & \xrightarrow{q_1} & \mathbb{R} \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle p=e^{2\pi it}} \\ X & \xrightarrow{f} & S^1 \end{array}$$

By the proposition above these covers have the same fibre as $\mathbb{R}$ over S^1, which is $\mathbb{Z}$, so they are certainly infinite cyclic covers. In fact, every infinite cyclic cover of a space is isomorphic to a pullback cover.

Fact

1. *Note that Mather's result implies that a finitely dominated space X is homotopy equivalent to a finite dimensional CW-complex. Namely, if $X \times S^1 \simeq K$ with K a finite CW-complex, then X is homotopy equivalent to an infinite cyclic covering space of K.*

2. *The answer to the question above is no. In 1965, de Lyra gave first examples of finitely dominated spaces which are not homotopy equivalent to a finite CW-complex. S. Ferry also gave a counter-example in 1980.*

Recall

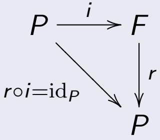
Let R be a ring. The following conditions on an R-module P are equivalent:

(a) *P is projective.*

(b) *P is a direct summand of a free module, i.e., $P \bigoplus Q \cong F$.*

(c) *There exists an $e \in \mathrm{End}_R(F)$ such that $P = e(P)(= e^2(P))$ and $Q = \ker e$*

(d) *There is a free module F and there are R-linear maps $i : P \to F$ and $r : F \to P$ with $r \circ i = 1_P$.*



Theorem (Systematic approach by C. T. C. Wall, 1965,1966)

A CW-complex X is finitely dominated (homotopy equivalent to a finite CW-complex) if and only if $\pi_1(X)$ is finitely presented and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$ is chain homotopy equivalent to a finite chain complex $\mathcal{P}$ of finitely generated projective(free) $\mathbb{Z}[\pi_1(X)]$-modules.

Sketch of proof: If X is dominated by a finite CW-complex K, then $\pi_1(X)$ is a retract of the finitely presented group $\pi_1(K)$ ($r_* \circ i_* = 1$), and is thus also finitely presented. Note that $C_*(\widetilde{X})$ is a chain homotopy direct summand of the finite finitely generated free $\mathbb{Z}[\pi_1(X)]$-module chain complex $\mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(K)]} C_*(\widetilde{K})$. It follows from the algebraic theory of Raniki (or by the original geometric argument of Wall) that $C_*(\widetilde{X})$ is chain homotopy equivalent to a finite finitely generated projective $\mathbb{Z}[\pi_1(X)]$-module chain complex $\mathcal{P}$.

Remark

Finitely dominated space $\Leftrightarrow$ *Finitely generated projective module*
Finite CW-complex $\Leftrightarrow$ *Finitely generated free module*

The difference between the homotopy types of finite and finitely dominated CW-complexes is precisely the difference between finitely generated projective and finitely generated free modules.

Theorem

Let S be a commutative semigroup (not necessarily having a unit). There is an abelian group G (called Grothendieck group or group completion of S), together with a semigroup homomorphism $\varphi : S \to G$, such that for any group H and homomorphism $\psi : S \to H$, there is a unique homomorphism $\theta : G \to H$ with $\psi = \theta \circ \varpi$. Uniqueness holds in the following sense: if $\varphi' : S \to G'$ is any other pair with the same property, then there is an isomorphism $\alpha : G \to G'$ with $\varphi' = \alpha \circ \varphi$.

Definition

Let R be a ring with unit. Then $K_0(R)$ is the Grothendieck group of the semi group $\mathrm{Proj}\, R$ of isomorphism classes of finitely generated projective modules over R, i.e., $K_0(R) := G(\mathrm{Proj}\, R)$.

Definition (Another definition)

For any ring R, the Grothendieck group $K_0(R)$ is the quotient of the free abelian group on isomorphism classes $[P]$ of finitely generated projective modules $P \in \mathcal{P}(R)$ by the subgroup generated by the elements of the form $[P \oplus Q] - [P] - [Q]$ for all P, Q in $\mathcal{P}(R)$.

Note

(a) K_0 *is a functor.*

(b) $K_0(R) = \mathbb{Z}$ *when* R *is a filed(or more generally a division ring), PID or local ring.*

Remark

For any ring R *with unit, there is a unique ring homomorphism* $i : \mathbb{Z} \to R$ *sending* 1 *to the unit of* R*. We obtain a map* $i_* : \mathbb{Z} \to K_0(R)$*. The image of* i_* *is the subgroup of* $K_0(R)$ *generated by the finitely generated free* R*-modules.*

Definition

The reduced projective class group $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

Remark

Let P be a finitely generated projective R-module. It is stably finitely generated free, i.e., $P \oplus R^m \cong R^n$ for $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$. Hence $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.

Spherical space form problem : Classify compact manifolds M^n having a sphere as universal cover $\widetilde{M}^n \cong S^n$.

Note

Let G be a finite group which admits a finite dimensional free G-CW-complex X homotopy equivalent to a sphere S^n, where n is odd. Then there are exact sequences of $\mathbb{Z}G$-modules

$$(*) \qquad 0 \to \mathbb{Z} = \frac{Z_n}{B_n} \to \frac{C_n}{B_n} \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0,$$

$$0 \to C_{\dim X} \to \cdots \to C_n \to \frac{C_n}{B_n} \to 0.$$

Note that each C_i is $\mathbb{Z}G$-free and $\frac{C_n}{B_n}$ is $\mathbb{Z}$-free. Since G is finite and $\mathrm{proj.dim}_{\mathbb{Z}G} \frac{C_n}{B_n} < \infty$, we know that $\frac{C_n}{B_n}$ is projective.

Splicing the resolution $(*)$, we conclude that G admits a following periodic projective resolution:

$$\cdots \to P_n \to \cdots \to P_0 \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

Thus G has periodic cohomology (after 0-step), i.e., $H^i(G)$ and $H^{i+n+1}(G)$ are naturally equivalent for all $i > 0$.

• If a finite group G acts freely on a sphere, then every abelian subgroups of G is cyclic (Smith, 1940).

Theorem

The following conditions are equivalent for a finite group G:

(a) *G has periodic cohomology, i.e., $H^i(G)$ and $H^{q+i}(G)$ are naturally equivalent for all $i > 0$.*

(b) *Every abelian subgroup of G is cyclic.*

(c) *Every elementary abelian p-subgroups of G has rank ≤ 1.*

(d) *The Sylow subgroups of G are cyclic or generalized quaternion groups.*

• Does every periodic group act freely on a sphere?

No!, as a consequence of a result due to Milnor(1957):
If a finite group G acts freely on a sphere, then every involution in G is central.
For example, the dihedral group D_{2p} cannot act freely on any sphere.

• A finite group G acts freely on a finite CW-complex X which is homotopy equivalent to a sphere if and only if every abelian subgroup of G is cyclic (Swan, 1960).

• A finite group G acts freely on some sphere if and only if G satisfies the both p^2 and the $2p$ condition for all primes p , i.e., every subgroup of order p^2 or $2p$ is cyclic (These are precisely the conditions founded by Smith and Milnor).
(Madsen, Thomas, and Wall, 1978).

Note

Swan showed the following:
Let G be a finite periodic group of period n. Then G admits periodic projective resolutions

$$0 \to \mathbb{Z} \to P_{kn-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0, \qquad k \geq 1,$$

with each P_i finitely generated projective. For every $k \geq 1$, $0 = s_{kn}(G) \in \widetilde{K}_0(\mathbb{Z}G)/T(\mathbb{Z}G)$ if and only if there is a finite CW-complex $X \simeq S^{kn-1}$ on which G acts freely. Also he showed that if G has period n, there exit k and a finite CW-complex $X \simeq S^{kn-1}$ on which G acts freely.
By Wall, it turns out that it suffices to take $k = 2$.

Definition

A complex X is called an *Eilenberg-Maclane complex* of type $(G, 1)$, or simply a $K(G, 1)$-complex if the following hold:

(1) X is connected.

(2) $\pi_1(X) = G$.

(3) The universal cover $\widetilde{X}$ of X is contractible.

Note

(1) *If X is $K(G, 1)$, then $C_*(\widetilde{X}) \twoheadrightarrow \mathbb{Z}$(the augmented cellular chain complex of the universal cover of X) is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ ($\Rightarrow \mathrm{gd}\, G \leq \mathrm{cd}\, G$).*

(2) *$K(G, 1) =$ Classifying space for free action, since $[X, K(G, 1) = BG] \leftrightarrow P(G, X)$(isomorphism classes of principle G-bundle), where X is a paracompact space.*

Definition

For a group G, let $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ be a projective resolution over $\mathbb{Z}G$. Then G is said to be of type

(1) FP_n if P_i is finitely generated for $0 \leq i \leq n$.

(2) FP_∞ if P_i is finitely generated for all i.

(3) FP if $P_* \to \mathbb{Z} \to 0$ is finite (i.e., the resolution has finite length and each P_i is finitely generated).

(4) FL if $P_* \to \mathbb{Z} \to 0$ is finite and each P_i is free.

Remark

A group G is of type FP if and only if G is of type FP_∞ and $\mathrm{cd}\, G := \inf\{n : H^i(G, -) = 0 \text{ for } i > n\} < \infty$.

The following theorem shows that the homotopy type of $K(G,1)$ is a topological counterpart of the groups of type *FP* or *FL*.

Theorem

If there exists a finitely dominated (resp. finite) $K(G,1)$, then G is of type FP (resp. FL).

Eilenberg, Ganea, and Wall showed that the converse of implication is true for the case $\mathrm{cd}\, G \geq 3$.

Theorem

Let G be an arbitrary group and let $n = \max\{\mathrm{cd}\, G, 3\}$. Then there exists an n-dimensional $K(G,1)$-complex X. If G is finitely presented and of type FL (resp. FP) then X can be taken to be finite (resp. finitely dominated).

Theorem (C. T. C. Wall, 1965,1966)

(a) *A finitely dominated space X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{w}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes.*

(b) *Given a finitely presented group G and $\widetilde{\sigma} \in \widetilde{K}_0(\mathbb{Z}G)$ $(\sigma \in K_0(\mathbb{Z}G))$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $\widetilde{w}(X) = \widetilde{\sigma}$. $(w(X) = \sigma)$.*

Here $w(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}[\pi_1(X)])$, where $C_*(\widetilde{X})$ is chain homotopy equivalent to a complex P_* of type FP over $\mathbb{Z}[\pi_1(X)]$ ($w(X)$ depends only on the homotopy type X).

Corollary

(a) *Every finitely dominated simply connected space is homotopy equivalent to a finite CW-complex (since $\widetilde{K}_0(\mathbb{Z}) = 0$).*

(b) *The following are equivalent for a finitely presented group G;*

   (1) *Every finitely dominated CW-complex with $\pi_1(X) \cong G$ is homotopy equivalent to a finite CW-complex.*

   (2) $\widetilde{K}_0(\mathbb{Z}G) = 0$.

Theorem (D.S.Rim, 1959)

Let $\pi = \langle x \rangle$ be a cyclic group of prime order p. Consider the map $\mathbb{Z}\pi \to \mathbb{Z}((\exp(2\pi i/p)))$ given by mapping x to the primitive p-th root of unity $\exp(2\pi i/p)$. Then the induced map of reduced projective class groups

$$\widetilde{K}_0(\mathbb{Z}\pi) \to \widetilde{K}_0(\mathbb{Z}(\exp(2\pi i/p)))$$

is an isomorphism.

Example

Let $G = \mathbb{Z}/p\mathbb{Z}$ with p a prime. Then

$$\widetilde{K}_0(\mathbb{Z}G) \cong \widetilde{K}_0(\mathbb{Z}(\exp(2\pi i/p))) \cong C(\mathbb{Z}(\exp(2\pi i/p)))$$

the ideal class group of the ring of algebraic integers $\mathbb{Z}(\exp(2\pi i/p))$ of the cyclotomic field of pth roots of unity $\mathbb{Q}(\exp(2\pi i/p))$ which is known to be trivial for all prime $p \leq 19$ and non-trivial for all other primes. Thus there are so many finitely dominated space which are not homotopy equivalent to a finite CW-complexes, i.e., for every prime $p \geq 23$, there exists a connected, finitely dominated CW-complex X with fundamental group $\mathbb{Z}/p\mathbb{Z}$ such that X is not homotopy equivalent to any finite CW-complex.

Conjecture

If X is a finitely dominated space such that $\pi_1(X)$ is torsion-free, then X is homotopy equivalent to a finite CW-complex.

Definition

A group G acts nilpotently on a $\mathbb{Z}G$-module M if there is a positive integer $n > 0$ such that $(I\mathbb{Z}G)^n M = 0$ (if and only if there exists a finite filtration $M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$ of M by $\mathbb{Z}G$-submodules such that M_i/M_{i+1} is a trivial $\mathbb{Z}G$-module.)

Definition

A path-connected space X is called nilpotent if $\pi_1(X)$ is nilpotent and for $n \geq 2$, $\pi_n(X)$ is a nilpotent $\pi_1(X)$-module.

Theorem

Let X be a nilpotent space. Then the following are equivalent:

(a) $\pi_i(X)$ *is finitely generated for all* $n \geq 1$.

(b) $H_i(X)$ *is finitely generated for all* $n \geq 1$.

(c) X *is homotopically equivalent to a CW-complex of finite type.*

Theorem (Mislin)

If X is a finitely dominated nilpotent space such that $\pi_1(X)$ is infinite (in this case $w(X) = 0$) or finite cyclic prime order, then X is homotopy equivalent to a finite CW-complex.

Conjecture

Every finitely dominated $K(G,1)$ space is homotopy equivalent to a finite CW-complex.

Conjecture

If G is of type FP, then G is of type FL.

Proposition

Let G be a group of type FP and let $0 \to P \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ be a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ with each F_i free. Then G is of type FL if and only if P is stably free.

Remark

The question whether there exist a group of type FP which is not of type FL has led to a more fundamental question: Do there exist finite generated projectives which are not stably free? Over a general ring the answer is YES (e.g., $\mathbb{Z}G$, where $G = \mathbb{Z}_{23}$). However there are no known examples with G torsion-free.

Definition

A G-CW-complex X is an $\underline{E}G$, or universal proper G-space, if X^H is contractible when $H < G$ is finite and X^H is empty otherwise. The minimal dimension of such an $\underline{E}G$ is denoted by $\underline{gd}G$.

Note

1. If G is torsion-free, then $EG = \underline{E}G$.
2. Let X be a proper G-CW-complex. Then up to G-homotopy, there is a unique G-map $X \to \underline{E}G$, i.e., X is a terminal object in the homotopy category of proper G-CW-complexes.

Note

(a) *Let $\mathcal{F}$ denote the set of finite subgroups of G. The orbit category $O_{\mathcal{F}}(G)$ has as objects the coset spaces G/K for $K \in \mathcal{F}$ and as morphism sets $\mathrm{mor}(G/K, G/L)$ the G-maps $G/H \to G/K$. The category $G\text{-}\mathrm{Mod}_{\mathcal{F}}$ has as objects the covariant functors $M : O_{\mathcal{F}}(G) \to \mathrm{Ab}$ and as morphisms the natural transformation.*

(b) *If G is torsion-free, then $O_{\mathcal{F}}(G)$ is naturally equivalent to the category of left $\mathbb{Z}G$-modules.*

(c) *$G\text{-}\mathrm{Mod}_{\mathcal{F}}$ has enough projectives.*

(d) Recall that $H_*(G,M) \cong H_*(K(G,1);\mathcal{M}) := H_*(\widetilde{K(G,1)} \otimes_{\mathbb{Z}G} M)$. Note that the chain complex $C_*(\underline{E}G)$ defines a projective resolution of the constant functor $\underline{\mathbb{Z}} \in G - \mathrm{Mod}_{\mathcal{F}}$. Thus $H^*_{\mathcal{F}}(\underline{E}G, M) \cong H^*_{\mathcal{F}}(G, M)$.

(e) The Eilenberg-Ganea conjecture is the statement that $\mathrm{gd}\, G = \mathrm{cd}\, G$. It is known that the conjecture is affirmative except for $\mathrm{cd}\, G = 2$ and $\mathrm{gd}\, G = 3$. Lück showed that $\underline{gd}\, G = \underline{cd}\, G$ except $\underline{cd}\, G = 2$ and $\underline{gd}\, G = 3$. However, there exists a group G with $\underline{gd}\, G = 3$ and $\underline{cd}\, G = 2$ (Brady-Leary,Nucinkis, 2001).

(f) Every finitely presented group is of type FP_2. In 1997, Bestvia and Brady showed that the converse is false. They also showed that either the Eilenberg-Ganea conjecture or the Whitehead conjecure (or possibly both) is false.

Note

Let L be a finite flag complex, i.e., L is a simplicial complex where every complete graph contained in $L^{(1)}$ is the 1-skeleton of a simplex in L (or any collection of $q+1$ mutually adjacent vertices span a q-simplex in L). The barycentric subdivision of any simplicial complex is a flag complex. The right-angled Artin group associated to L has a presentation where the generators correspond to the vertices L, and the defining relations state that two generators commute when their affiliated vertices are joined by an edge. Denote this group by G_L. For example, in case when L is the n-simplex, $G_L \cong \bigoplus^{n+1} \mathbb{Z}$, and in case when L consists of n-vertices and no edges, $G_L \cong *^n \mathbb{Z}$.

Theorem

Let L be a finite flag complex, let G_L be the associated right-angled Artin group, and let H_L be the kernel of the map sending every generator of G_L to $1 \in \mathbb{Z}$. Then the following hold:

(a) *H_L is FP_n if and only if L is n-acyclic.*

(b) *H_L is finitely presented if and only if L is simply connected.*

Remark

It follows that if L is not simply connected, but has trivial reduced homology in dimensions 1 and 2, then H_L will be FP_2 but it is not finitely presented.

Theorem (Kropholler, Martinez-Perez, Nucinkis)

Every elementary amenable group of type FP is of type F, i.e., there exists a finite $K(G, 1)$-complex. (Since $F \Rightarrow FL$, every elementary amenable group of type FP is of type FL).

Remark

1. *It was well-known that every nilpotent group of type FP is of type F.*
2. *The class of elementary amenable group $\mathcal{EAM}$ is the smallest class of groups with the following properties:*
   - (a) *$\mathcal{EAM}$ contains all finite groups and all abelian groups.*
   - (b) *$\mathcal{EAM}$ is closed under extensions.*
   - (c) *$\mathcal{EAM}$ is closed under upwards directed unions.*

Conjecture (Projective class group conjecture)

If G is torsion-free, then $\widetilde{K}_0(\mathbb{Z}G) = 0$.

It is known that Projective class group conjecture holds for free groups, free abelian groups, Bieberbach groups, virtually Poly-$\mathbb{Z}$-groups, fundamental groups of closed Riemmann manifolds all of whose sectional curvature values are non-positive.

Lemma (Whitehead Lemma)

For any ring R, $[GL(R), GL(R)] = E(R)$ *and* $[E(R), E(R)] = [E(R)]$, *i.e.,* $E(R)$ *is perfect or* $E(R)_{ab}$ *is trivial or* $H_1(E(R), \mathbb{Z}) = 0$.

Definition

For any ring R,
$K_1(R) := GL(R)/E(R) = GL(R)_{ab} \cong H_1(GL(R), \mathbb{Z})$.

Definition

If G is a group, its Whitehead group $Wh(G)$ is the quotient of $K_1(\mathbb{Z}G)$ by the image of $\{\pm g \mid g \in G\} \subseteq (\mathbb{Z}G)^{\times}$.

Definition (h-corbordism)

An h-cobordism over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \coprod M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (s-cobordism Theorem)

1. *Let M_0 be a closed (smooth) manifold dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0. Then W is homeomorphic(diffeomorphic) to $M_0 \times [0,1]$ relative M_0 if and only if its whitehead torsion $\tau(W, M_0) \in Wh(\pi_1(M_0))$ vanishes.*
2. *Let G be a finitely presented group G, n an integer $n \geq 5$ and x an element in $Wh(G)$. Then there exists an n-dimensional h-corbordism $(W; M_0, M_1)$ over M_0 with $\tau(W, M_0) = x$.*

Corollary (Smale)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$:

(a) *Every compact n-dimensional h-corbordism W with $\pi_1(M) \cong G$ is trivial.*

(b) $Wh(G) = \{0\}$.

Corollary

For $n \geq 5$, the Poincarè conjecture is true.

Conjecture (Whitehead torsion conjecture)

If G is torsion-free, then $Wh(G) = 0$.

Theorem (Bass-Heller-Swan)

For any group G,

$$Wh(G \times \mathbb{Z}) \cong Wh(G) \bigoplus Nil(\mathbb{Z}G) \bigoplus Nil(\mathbb{Z}G) \bigoplus \widetilde{K}_0(\mathbb{Z}G).$$

Whitehead torsion conjecture implies Projective class group conjecture, since $G \times \mathbb{Z}$ is torsion-free whenever G is torsion-free.

For a map $\phi : L \to Y$, let $M_\phi = Y \cup_\phi (L \times I)$ its mapping cylinder. We define $\pi_n(\phi) = \pi_n(M_\phi, L \times 1)$, and call ϕ *n-connected* if L and Y are connected and $\pi_i(\phi) = 0$ for $1 \leq i \leq n$.

- $F1$: The group $\pi_1(Y)$ is finitely generated.
- $F2$: The group $\pi_1(Y)$ is finitely presented, and for all finite 2-complexes L and map $\phi : L \to Y$ inducing an isomorphism $\phi_* : \pi_1(L) \to \pi_1(Y)$, $\pi_2(\phi)$ is a finitely generated $\mathbb{Z}\pi_1(Y)$-module.
- Fn $(n \geq 3)$: Condition $F(n-1)$ holds, and for all finite $(n-1)$-complexes L and $(n-1)$-connected map $\phi : L \to Y$, $\pi_n(\phi)$ is a finitely generated $\mathbb{Z}\pi_1(Y)$-module.
- Dn : $H_i(\widetilde{Y}) = 0$ for $i > n$, and $H^{n+1}(Y; \mathcal{M}) = 0$ for all coefficient system $\mathcal{M}$ over Y.

Theorem

Let Y be a connected complex. Then the following hold:

(1) *Y satisfies Fn if and only if it is homotopy equivalent to a complex with finite n-skeleton.*

(2) *Y satisfies Dn $(n \neq 2)$ if and only if it is homotopy equivalent to an n-dimensional complex.*

(3) *For $n \geq 3$, Y is dominated by a finite complex of dimension n if and only if Y satisfies Fn and Dn.*

D(2)-Problem : Let X be a finite connected 3-complex, and suppose that $H_3(\widetilde{X}) = H^3(X;\mathcal{M}) = 0$ for all local coefficient system $\mathcal{M}$ over X. Is it true that X is homotopy equivalent to a finite complex of dimension 2?

Realization-problem : Let G be a finitely presented group. Is every algebraic 2-complex is geometrically realizable? i.e., there is a finite 2-complex X with $\pi_1(X) = G$, and an algebraic chain homotopy equivalence $\phi : C_*(X) \cong E$:

$$E : 0 \to J \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

where F_i is finitely generated stably free $\mathbb{Z}G$-module and $C_*(X) := 0 \to \pi_2(X) \to C_2(\widetilde{X}) \to C_1(\widetilde{X}) \to C_0(\widetilde{X}) \to \mathbb{Z} \to 0.$

Theorem (F.E.A. Johnson, 2003 (7 p.m, 19, June, 1969))

(a) *Let G be a finite group. The $D(2)$-property holds for G if and only if each algebraic 2-complex over G is geometrically realizable.*

(a) *The $D(2)$-property holds for all non-abelian free group of finite rank.*

Recall that

$$\chi(G) = \sum(-1)^i b_i(G) = \sum(-1)^i \dim_{\mathbb{Q}}(H_i(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q})$$

if it is well-defined.
Suppose that we embed $\mathbb{Z}G$ in a ring R(or even more generally, we have a homomorphism from $\mathbb{Z}G$ to R) and there is a nice dimension function defined on R-modules. Then we can define Betti numbers and Euler Characteristics with respect to R.

Note (Trace functions)

Let S be an R-algebra and M an R-module. An R-linear map $f : S \to M$ is called a trace if $f(ab) = f(ba)$ for all $a, b \in S$. There is a universal trace $T_{uni} : S \to S/[S, S]$, where $[S, S] \subset S$ denote the R-module generated by the elements of the form $ab - ba$ in S. Every trace $S \to M$ factors uniquely through T_{uni}. Any trace $t : S \to M$ gives rise to a trace $M_n(t) : M_n(S) \to M$ by putting $M_n(t)(A) = \sum_i t(a_{ii})$, where $A = (a_{ij}) \in M_n(S)$.

Note (Trace functions)

One can then define the trace $t(\phi)$ for any S-map $\phi : F \to F$, F a finitely generated free S-module by choosing an S-basis of F and putting $t(\phi) = M_n(t)(A(\phi))$, where $A(\phi)$ is the matrix of ϕ with respect to the chosen basis. Because of the trace property of t, $t(\phi)$ is well-defined. To define the trace of a finitely generated projective S-module P one represents P as the image of an idempotent S-map $\phi_P : S^n \to S^n$ and one puts $t(P) := t(\phi_P)$; it only depends on the isomorphism class of P. One therefore obtains a homomorphism of abelian group $t : K_0^{alg}(S) \to M$.

Definition

Let $l^2(G)$ be the Hilbert space of square summable functions $f : G \to \mathbb{C}$, which we also denote by $f = \sum_{g\in G} f(g)\cdot g$ with $\sum_{g\in G} |f(g)|^2 < \infty$, i.e., $l^2(G) = \{f : G \to \mathbb{C} \mid \sum_{g\in G} |f(g)|^2 < \infty\}$. Define a scalar product as follows: for $f, h \in l^2(G)$, $\langle f, h\rangle := \sum_{g\in G} f(g)\overline{h(g)}$. Denote by δ_g the element of $l^2(G)$ taking value 1 in $g \in G$ and 0 otherwise. $\{\delta_g\}$ is called Hilbert basis, as any element of $l^2(G)$ can be expressed as an infinite linear combination of the δ_g's, with square summable coefficients. It is well-known that $l^2(G)$ is the same as the Hilbert space completion of the complex group ring $\mathbb{C}G$ with respect to the pre-Hilbert space structure for which G is an orthonormal basis. The group von Neumann algebra $\mathcal{N}(G)$ is the C^*-algebra $B(l^2(G))^G$ of G-equivariant bounded operators from $l^2(G)$ to $l^2(G)$.

Definition

The von Neumann trace on $\mathcal{N}(G)$ is defined by

$$tr_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \;\; f \to \langle f(e), e \rangle_{l^2(G)},$$

where $e \in G \subset l^2(G)$ is the unit element.
For any finitely generated projective $\mathbb{Z}G$-module P, let $M = (m_{ij}) \in M_n(\mathbb{Z}G)$ be the associated idempotent matrix, i.e., P is isomorphic to the image under the right multiplication $\mathbb{Z}G^n \to \mathbb{Z}G^n$ by M.
For any $\mathcal{N}(G)$-module M, $\dim_{\mathcal{N}(G)}(M)$ be the extended von Neumann dimension function which takes real values in $[0, \infty]$.

Definition

For an arbitrary G-space X, its p-th L^2-Betti number is defined by

$$b_p^{(2)}(X) := \dim_{\mathcal{N}(G)}(H_p^G(X, \mathcal{N}(G))),$$

where $H_p^G(X, \mathcal{N}(G))$ is the homology of the $\mathcal{N}(G)$-chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)$.
The L^2-Euler characteristic of an arbitrary G-space X is defined by

$$\chi^{(2)}(X; G) = \chi^{(2)}(X) := \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(X)$$

provided that $h^{(2)}(X) := \sum_{p \geq 0} b_p^{(2)}(X) < \infty$. We define for any discrete group G its p-th L^2-Betti number by $b_p^{(2)}(G) := b_p^{(2)}(EG)$. The L^2-Euler characteristic of G is defined by $\chi^{(2)}(G) := \chi^{(2)}(EG)$ provided that $h^{(2)}(G) := h^{(2)}(EG) < \infty$. We convention that if X is a connected CW-complex with $G = \pi_1(X)$, then we denote $\chi^{(2)}(X) := \chi^{(2)}(\widetilde{X} : G)$.

Note

Let G be a discrete group and $\mathbb{Z}G$ its integral group ring. Denote by $[g]$ the conjugacy class of $g \in G$, by $[G]$ the set of conjugacy classes of elements of G, and by $[\mathbb{Z}G, \mathbb{Z}G]$ the additive subgroup of $\mathbb{Z}G$ generated by all commutators $gh - hg, (g, h \in G)$. We can identify $\mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G]$ with $\bigoplus_{[s]\in[G]} \mathbb{Z} \cdot [s]$. Recall the following three well-known trace functions. The Kaplansky trace is defined by

$$\kappa : \mathbb{Z}G \to \mathbb{Z}, \quad a = \sum a_g g \mapsto a_e,$$

and the augmentation trace is defined by

$$\epsilon : \mathbb{Z}G \to \mathbb{Z}, \quad a = \sum a_g g \mapsto \sum_{g \in G} a_g.$$

The *Hattori-Stallings trace* is defined by

$$HS : \mathbb{Z}G \to \bigoplus_{[s]\in[G]} \mathbb{Z}\cdot[s]$$

$$a = \sum a_g g \mapsto a + [\mathbb{Z}G, \mathbb{Z}G] = \sum_{[s]\in[G]} \epsilon_{[s]}(a)[s],$$

where for $[s] \in [G]$, $\epsilon_{[s]}(a) = \sum_{g\in[s]} a_g$ is a partial augmentation.

It was well-known that for each trace function, there are well-defined trace maps on $K_0(\mathbb{Z}G)$. By a slight abuse of notation, we also denote $\kappa : K_0(\mathbb{Z}G) \to \mathbb{Z}$, $\epsilon : K_0(\mathbb{Z}G) \to \mathbb{Z}$, and $HS : K_0(\mathbb{Z}G) \to \bigoplus_{[s]\in[G]} \mathbb{Z} \cdot [s]$ for their induced maps.
Note that for any finitely generated projective $\mathbb{Z}G$-module P,

$$\epsilon(P) = \mathrm{rk}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P),$$

Note also that if G is finite, then for any finitely generated projective $\mathbb{Z}G$-module P,

$$\kappa(P) = \mathrm{rk}_{\mathbb{Z}}(P)/|G|.$$

In 1976, Bass conjectured that following conjectures:

Strong Bass Conjecture

For any group G, the map $HS : K_0(\mathbb{Z}G) \to \bigoplus_{[s]\in[G]} \mathbb{Z} \cdot [s]$ has image in $\mathbb{Z} \cdot [e]$.

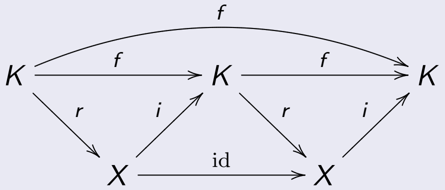
Weak Bass Conjecture

For any group G, the maps $\epsilon, \kappa : K_0(\mathbb{Z}G) \to \mathbb{Z}$ coincide.

Note that if the strong Bass conjecture is a theorem, then so is the weak Bass conjecture.

Note

Let X be a finitely dominated space, i.e. there exists a finite complex K and maps $i : X \to K$ and $r : K \to X$ with $r\circ \simeq \mathrm{id}_X$. Then $(i\circ r)^2 = i\circ r\circ i\circ r = i\circ (r\circ i)\circ r \simeq i\circ r$, i.e, $i\circ r$ is homotopy idempotent. Conversely, suppose that a self map $f : K \to K$ on a finite CW-complex K is homotopy idempotent. By Hastings and Heller(every homotopy idempotent map on a finite dimensional complex split), there exists a CW-complex X and maps $i : X \to K$ and $r : K \to X$ such that $r\circ i \simeq \mathrm{id}_X$ and $i\circ r \simeq f$.



Theorem (Geoghegan, 1981)

For a finitely presented group G, the following are equivalent:

(a) *The strong Bass conjecture holds for G*

(b) *Every homotopy idempotent selfmap f on a finite connected complex with fundamental group G has Nielsen number either zero or one.*

Theorem (Berrick, Chatterji, and Mislin, 2006)

For a finitely presented group G, the following are equivalent:

(a) *The strong Bass conjecture holds for G*

(b) *Every homotopy idempotent selfmap of closed, smooth and oriented manifold of dimension greater than 2 is homotopic to one that has precisely one fixed point.*

Remark

1. *Since the fundamental group G of a closed, smooth and oriented 3-manifold M satisfies the strong Bass conjecture, the statement (b) above holds.*

2. *For F a closed surface of negative Euler characteristic and $n \geq 2$, Kelly(2006) constructs a homotopy idempotent selfmap $f_n : F \to F$ such that every map homotopic to f_n has at least n fixed points. On the other hand, the fundamental groups of surfaces are well-known to satisfy the strong Bass conjecture.*

Theorem (Berrick, Chatterji, and Mislin, 2006)

For a finitely presented group G, the following are equivalent:

(a) *The weak Bass conjecture holds for G*

(b) *For any finitely dominated CW-complex X with $\pi_1(X) = G$, we have $\chi^{(2)}(X) = \chi(X)$*

(c) *Every pointed homotopy idempotent selfmap of closed, smooth and oriented manifold M with $\pi_1(M) = G$ and inducing the identity on G has its Lefschetz number equal to the L^2-Lefschetz number of the induced G-map on the universal cover of M.*

Note

(a) *For any ring R, $K_2(R) \cong H_2(E(R), \mathbb{Z})$.*

(b) *The correspondence $G \to BG$ and $X \to \pi_1(X)$ fulfills $\pi_1(BG) \cong G$, but it is in general not true that the classifying space of the fundamental group of a space is homotopy equivalent to X.*

Quillen introduced a new way to go from group theory to topology and vice versa.

Theorem (Quillen's plus constuction)

Let X be a connected CW-complex, P a perfect normal subgroup of its fundamental group $\pi_1(X)$. There exists a CW-complex X_P^+, obtained from X by attaching 2-cells and 3-cells, such that the inclusion $i : X \to^{+}_{P X}$ satisfies the following:

(a) $i_* : \pi_1(X) \to \pi_1(X_P^+)$ *is exactly the quotient map* $\pi_1(X) \twoheadrightarrow \pi_1(X)/P$.

(b) i *induces an isomorphism* $i_* : H_*(X; A) \to H_*(X_P^+; A)$ *for any local coefficient system* A *on* X_P^+.

(c) X_P^+ *is unique up to homotopy equivalence.*

Theorem (Kan-Thurston, 1976)

For every connected CW-complex X, there is a group G_X and a map $t_X : BG_X = K(G_X, 1) \to X$, natural for maps of X, with the following properties:

(a) *the map $(t_X)_* : \pi_1(BG_X) \cong G_X \to \pi_1(X)$ is surjective (and the kernel P_X of $(t_X)_*$ is perfect.*

(b) *the map t_X induces isomorphisms $(t_X)_* : H_*(BG_X; A) \to H_*(X; A)$ for every local coefficient system A on X.*

Corollary

For every connected CW-complex X, there exists a group G_X together with a perfect normal subgroup P_X such that one has a homotopy equivalence $(BG_X)^+_{P_X} \simeq X$.

Remark

1. *Equivalence classes of topogenic groups*
   $\leftrightarrow$ *Homotopy types of CW-complexes*
   $$(G, P) \to BG^+_P$$
   $$(G_X, P_X) \leftarrow X.$$
2. *For any ring R and any positive integer i, the i-th algebraic K-theory group of R is defined by $K_i(R) := \pi_i(BGL(R)^+_{E(R)})$.*

Moreover, Baumslag, Dyer, and Heller asserted that if X is a finite simplicial complex, then the space BG_X above can be taken to be a finite simplicial complex of the same dimension as X. Notice that it is sufficient to consider only finitely presented groups considering Bass conjectures. Recall that a group G is finitely presented and of type FP (respectively, FL) if and only if G admits a finitely dominated (respectively, finite) $K(G,1)$-complex. Thus, if G is a finitely presented group of type FL, then $\chi^{(2)}(G) = \chi(G)$ by Atiyah's L^2-Index theorem (1976). In this view point, we may expect that the following conjecture is true and its validity is equivalent to that of the weak Bass conjecture.

Lemma

Let X be a finitely dominated space. Then there exists a group G_X of type FP such that G_X satisfy the properties of Kan-Thurston construction.

Conjecture A

Let G be a group of type FP. Then $\chi^{(2)}(G) = \chi(G)$.

Remark

In 1996, Eckmann showed that every group G of type FP satisfies $\chi^{(2)}(G) = \chi(G)$ if G fulfills the strong Bass conjecture.

Theorem

The following statements are equivalent:

(a) *Conjecture A is a theorem.*

(b) *The weak Bass conjecture is a theorem.*

Corollary

Let G be a group of type FP. If G is amenable, then $\chi(G) = 0$.