Open Problems on Critical Graphs

Tommy R. Jensen

Department of Mathematics
Kyungpook National University

2009 Combinatorics Workshop
KAIST, August 20–21, 2009

Definitions

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

- A k-coloring of G is a map $f : V \to C$ with $|C| = k$, such that $f(u) \neq f(v)$ holds for every edge $uv \in E$.
- G is k-chromatic, written $\chi(G) = k$, if k is the least number such that a k-coloring of G exists.
- G is critical if $\chi(H) < \chi(G)$ holds for every proper subgraph H of G. Equivalently, $G = K_1$ or $\chi(G - e) < \chi(G)$ for all $e \in E$.
- G is vertex-critical if $\chi(H) < \chi(G)$ holds for every proper induced subgraph H of G. Equivalently, $\chi(G - v) < \chi(G)$ for all $v \in V$.

Definitions

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

- A *k*-coloring of G is a map $f : V \to C$ with $|C| = k$, such that $f(u) \neq f(v)$ holds for every edge $uv \in E$.
- G is *k*-chromatic, written $\chi(G) = k$, if k is the least number such that a k-coloring of G exists.
- G is critical if $\chi(H) < \chi(G)$ holds for every proper subgraph H of G. Equivalently, $G = K_1$ or $\chi(G - e) < \chi(G)$ for all $e \in E$.
- G is vertex-critical if $\chi(H) < \chi(G)$ holds for every proper induced subgraph H of G. Equivalently, $\chi(G - v) < \chi(G)$ for all $v \in V$.

Definitions

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

- A *k-coloring* of G is a map $f : V \to C$ with $|C| = k$, such that $f(u) \neq f(v)$ holds for every edge $uv \in E$.
- G is *k-chromatic*, written $\chi(G) = k$, if k is the least number such that a k-coloring of G exists.
- G is *critical* if $\chi(H) < \chi(G)$ holds for every proper subgraph H of G. Equivalently, $G = K_1$ or $\chi(G - e) < \chi(G)$ for all $e \in E$.
- G is *vertex-critical* if $\chi(H) < \chi(G)$ holds for every proper induced subgraph H of G. Equivalently, $\chi(G - v) < \chi(G)$ for all $v \in V$.

Definitions

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

- A k-coloring of G is a map $f : V \to C$ with $|C| = k$, such that $f(u) \neq f(v)$ holds for every edge $uv \in E$.
- G is k-chromatic, written $\chi(G) = k$, if k is the least number such that a k-coloring of G exists.
- G is critical if $\chi(H) < \chi(G)$ holds for every proper subgraph H of G. Equivalently, $G = K_1$ or $\chi(G - e) < \chi(G)$ for all $e \in E$.
- G is vertex-critical if $\chi(H) < \chi(G)$ holds for every proper induced subgraph H of G. Equivalently, $\chi(G - v) < \chi(G)$ for all $v \in V$.

Definitions

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

- A *k-coloring* of G is a map $f : V \to C$ with $|C| = k$, such that $f(u) \neq f(v)$ holds for every edge $uv \in E$.
- G is *k-chromatic*, written $\chi(G) = k$, if k is the least number such that a k-coloring of G exists.
- G is critical if $\chi(H) < \chi(G)$ holds for every proper subgraph H of G. Equivalently, $G = K_1$ or $\chi(G - e) < \chi(G)$ for all $e \in E$.
- G is vertex-critical if $\chi(H) < \chi(G)$ holds for every proper induced subgraph H of G. Equivalently, $\chi(G - v) < \chi(G)$ for all $v \in V$.

Examples

- $k < 3$
  The complete graph K_k is the unique critical k-chromatic graph.
- $k = 3$
  The critical (vertex-critical) graphs are precisely the odd cycles C_{2n+1} for $n \geq 1$ (König 1916).
- $k \geq 4$
  - As before, K_k is a critical k-chromatic graph.
  - But the number of non-isomorphic critical k-chromatic graphs of order n is at least c^{n^2}, for some $c > 1$ (V. Rödl).
  - And a vertex-critical graph is not necessarily critical.
  - Each decision problem CRITICAL k-CHROMATIC and VERTEX-CRITICAL k-CHROMATIC is an element of NP only if co-NP=NP: They are **hard** problems.

Examples

- $k < 3$
  The complete graph K_k is the unique critical k-chromatic graph.
- $k = 3$
  The critical (vertex-critical) graphs are precisely the odd cycles C_{2n+1} for $n \geq 1$ (König 1916).
- $k \geq 4$
  - As before, K_k is a critical k-chromatic graph.
  - But the number of non-isomorphic critical k-chromatic graphs of order n is at least c^{n^2}, for some $c > 1$ (V. Rödl).
  - And a vertex-critical graph is not necessarily critical.
  - Each decision problem CRITICAL k-CHROMATIC and VERTEX-CRITICAL k-CHROMATIC is an element of NP only if co-NP=NP: They are **hard** problems.

Examples

- $k < 3$
  The complete graph K_k is the unique critical k-chromatic graph.
- $k = 3$
  The critical (vertex-critical) graphs are precisely the odd cycles C_{2n+1} for $n \geq 1$ (König 1916).
- $k \geq 4$
  - As before, K_k is a critical k-chromatic graph.
  - But the number of non-isomorphic critical k-chromatic graphs of order n is at least c^{n^2}, for some $c > 1$ (V. Rödl).
  - And a vertex-critical graph is not necessarily critical.
  - Each decision problem CRITICAL k-CHROMATIC and VERTEX-CRITICAL k-CHROMATIC is an element of NP only if co-NP=NP: They are **hard** problems.

Examples

- $k < 3$
  The complete graph K_k is the unique critical k-chromatic graph.
- $k = 3$
  The critical (vertex-critical) graphs are precisely the odd cycles C_{2n+1} for $n \geq 1$ (König 1916).
- $k \geq 4$
  - As before, K_k is a critical k-chromatic graph.
  - But the number of non-isomorphic critical k-chromatic graphs of order n is at least c^{n^2}, for some $c > 1$ (V. Rödl).
  - And a vertex-critical graph is not necessarily critical.
  - Each decision problem CRITICAL k-CHROMATIC and VERTEX-CRITICAL k-CHROMATIC is an element of NP only if co-NP=NP: They are **hard** problems.

Examples

- $k < 3$
  The complete graph K_k is the unique critical k-chromatic graph.
- $k = 3$
  The critical (vertex-critical) graphs are precisely the odd cycles C_{2n+1} for $n \geq 1$ (König 1916).
- $k \geq 4$
  - As before, K_k is a critical k-chromatic graph.
  - But the number of non-isomorphic critical k-chromatic graphs of order n is at least c^{n^2}, for some $c > 1$ (V. Rödl).
  - And a vertex-critical graph is not necessarily critical.
  - Each decision problem CRITICAL k-CHROMATIC and VERTEX-CRITICAL k-CHROMATIC is an element of NP only if co-NP=NP: They are **hard** problems.

Examples

- $k < 3$
  The complete graph K_k is the unique critical k-chromatic graph.
- $k = 3$
  The critical (vertex-critical) graphs are precisely the odd cycles C_{2n+1} for $n \geq 1$ (König 1916).
- $k \geq 4$
  - As before, K_k is a critical k-chromatic graph.
  - But the number of non-isomorphic critical k-chromatic graphs of order n is at least c^{n^2}, for some $c > 1$ (V. Rödl).
  - And a vertex-critical graph is not necessarily critical.
  - Each decision problem CRITICAL k-CHROMATIC and VERTEX-CRITICAL k-CHROMATIC is an element of NP only if co-NP=NP: They are **hard** problems.

What are they for?

- Introduced by Dirac $\sim$ 1949. Discussed by Dirac and Erdős.
- Every critical graph is finite (de Bruijn & Erdős, 1951).
- Critical graphs have special structure: e.g. $\delta \geq \chi - 1$.
- Finding a (perhaps small) critical k-chromatic subgraph of G gives a certificate for $\chi(G) \geq k$.

What are they for?

- Introduced by Dirac $\sim$ 1949. Discussed by Dirac and Erdős.
- Every critical graph is finite (de Bruijn & Erdős, 1951).
- Critical graphs have special structure: e.g. $\delta \geq \chi - 1$.
- Finding a (perhaps small) critical k-chromatic subgraph of G gives a certificate for $\chi(G) \geq k$.

What are they for?

- Introduced by Dirac $\sim$ 1949. Discussed by Dirac and Erdős.
- Every critical graph is finite (de Bruijn & Erdős, 1951).
- Critical graphs have special structure: e.g. $\delta \geq \chi - 1$.
- Finding a (perhaps small) critical k-chromatic subgraph of G gives a certificate for $\chi(G) \geq k$.

What are they for?

- Introduced by Dirac $\sim$ 1949. Discussed by Dirac and Erdős.
- Every critical graph is finite (de Bruijn & Erdős, 1951).
- Critical graphs have special structure: e.g. $\delta \geq \chi - 1$.
- Finding a (perhaps small) critical k-chromatic subgraph of G gives a certificate for $\chi(G) \geq k$.

How many edges can they have?

Let $F_k(n)$ be the maximal number of edges of a critical k-chromatic graph of order n.

- Does $\lim F_k(n)/n^2$ exist?
- Is $F_6(n) = \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$?
- Does a constant $\varepsilon > 0$ exist such that $F_4(n) \geq (\frac{1}{16} + \varepsilon)n^2$ for infinitely many values of n?

P. Erdős 1949
G.A. Dirac 1952
B. Toft 1970

How many edges can they have?

Let $F_k(n)$ be the maximal number of edges of a critical k-chromatic graph of order n.

- Does $\lim F_k(n)/n^2$ exist?
- Is $F_6(n) = \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$?
- Does a constant $\varepsilon > 0$ exist such that $F_4(n) \geq (\frac{1}{16} + \varepsilon)n^2$ for infinitely many values of n?

P. Erdős 1949
G.A. Dirac 1952
B. Toft 1970

How many edges can they have?

Let $F_k(n)$ be the maximal number of edges of a critical k-chromatic graph of order n.

- Does $\lim F_k(n)/n^2$ exist?
- Is $F_6(n) = \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$?
- Does a constant $\varepsilon > 0$ exist such that $F_4(n) \geq (\frac{1}{16} + \varepsilon)n^2$ for infinitely many values of n?

P. Erdős 1949
G.A. Dirac 1952
B. Toft 1970

How many edges can they have?

Let $F_k(n)$ be the maximal number of edges of a critical k-chromatic graph of order n.

- Does $\lim F_k(n)/n^2$ exist?
- Is $F_6(n) = \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$?
- Does a constant $\varepsilon > 0$ exist such that $F_4(n) \geq (\frac{1}{16} + \varepsilon)n^2$ for infinitely many values of n?

P. Erdős 1949
G.A. Dirac 1952
B. Toft 1970

How many edges can they have?

Let $F_k(n)$ be the maximal number of edges of a critical k-chromatic graph of order n.

- Does $\lim F_k(n)/n^2$ exist?
- Is $F_6(n) = \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$?
- Does a constant $\varepsilon > 0$ exist such that $F_4(n) \geq (\frac{1}{16} + \varepsilon)n^2$ for infinitely many values of n?

P. Erdős 1949
G.A. Dirac 1952
B. Toft 1970

How many edges can they have?

- $F_6(n) \geq \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$.

  G.A. Dirac 1952

- $F_k(n) > c_k n^2$ when $k \geq 4$, where $c_4 \geq 1/16$ and $c_5 \geq 4/31$.

  B. Toft 1970

- $c_k \leq \frac{k-2}{2(k-1)}$.

  T.R.J. 2002

How many edges can they have?

- $F_6(n) \geq \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$.

  G.A. Dirac 1952

- $F_k(n) > c_k n^2$ when $k \geq 4$, where $c_4 \geq 1/16$ and $c_5 \geq 4/31$.

  B. Toft 1970

- $c_k \leq \frac{k-2}{2(k-1)}$.

  T.R.J. 2002

How many edges can they have?

- $F_6(n) \geq \frac{1}{4}n^2 + n$ for all $n \equiv 2 \pmod 4$.
  G.A. Dirac 1952
- $F_k(n) > c_k n^2$ when $k \geq 4$, where $c_4 \geq 1/16$ and $c_5 \geq 4/31$.
  B. Toft 1970
- $c_k \leq \frac{k-2}{2(k-1)}$.
  T.R.J. 2002

How few edges can they have?

Let $f_k(n)$ be the minimal number of edges of a critical k-chromatic graph of order n, where $k \geq 4$ and $n \geq k+2$.

- What is a best possible lower bound on $f_k(n)$?
- Does equality hold in $f_4(n) \leq \lfloor 5n/3 \rfloor$?
- Determine $R(k, s)$ such that

$$2|E| \geq (k-1)|V| + R(k, s)$$

is true if $K_s \not\subseteq G$.

G.A. Dirac 1957
B. Toft 1974

How few edges can they have?

Let $f_k(n)$ be the minimal number of edges of a critical k-chromatic graph of order n, where $k \geq 4$ and $n \geq k+2$.

- What is a best possible lower bound on $f_k(n)$?
- Does equality hold in $f_4(n) \leq \lfloor 5n/3 \rfloor$?
- Determine $R(k,s)$ such that

$$2|E| \geq (k-1)|V| + R(k,s)$$

is true if $K_s \not\subseteq G$.

G.A. Dirac 1957
B. Toft 1974

How few edges can they have?

Let $f_k(n)$ be the minimal number of edges of a critical k-chromatic graph of order n, where $k \geq 4$ and $n \geq k+2$.

- What is a best possible lower bound on $f_k(n)$?
- Does equality hold in $f_4(n) \leq \lfloor 5n/3 \rfloor$?
- Determine $R(k,s)$ such that

$$2|E| \geq (k-1)|V| + R(k,s)$$

is true if $K_s \not\subseteq G$.

G.A. Dirac 1957
B. Toft 1974

How few edges can they have?

Let $f_k(n)$ be the minimal number of edges of a critical k-chromatic graph of order n, where $k \geq 4$ and $n \geq k + 2$.

- What is a best possible lower bound on $f_k(n)$?
- Does equality hold in $f_4(n) \leq \lfloor 5n/3 \rfloor$?
- Determine $R(k, s)$ such that

$$2|E| \geq (k-1)|V| + R(k, s)$$

is true if $K_s \not\subseteq G$.

G.A. Dirac 1957
B. Toft 1974

How few edges can they have?

Let $f_k(n)$ be the minimal number of edges of a critical k-chromatic graph of order n, where $k \geq 4$ and $n \geq k+2$.

- What is a best possible lower bound on $f_k(n)$?
- Does equality hold in $f_4(n) \leq \lfloor 5n/3 \rfloor$?
- Determine $R(k, s)$ such that

$$2|E| \geq (k-1)|V| + R(k, s)$$

is true if $K_s \not\subseteq G$.

G.A. Dirac 1957
B. Toft 1974

How few edges can they have?

- $2f_k(n) \geq (k-1+(k-3)/((k-c)(k-1)+k-3))n$ for $k \geq 6$, where $c = (k-5)(1/2-1/(k-1)(k-2))$.
  A.V. Kostochka & M. Stiebitz 2003
- The constant $(k-1)$ in $2|E| \geq (k-1)|V| + R(k,s)$ has been improved.
  M. Krivelevich 1998

How few edges can they have?

- $2f_k(n) \geq (k-1+(k-3)/((k-c)(k-1)+k-3))n$ for $k \geq 6$, where $c = (k-5)(1/2 - 1/(k-1)(k-2))$.
  A.V. Kostochka & M. Stiebitz 2003
- The constant $(k-1)$ in $2|E| \geq (k-1)|V| + R(k,s)$ has been improved.
  M. Krivelevich 1998

Minimal degrees and regularity.

Let $\delta_k(n)$ be the largest minimal degree of a critical k-chromatic graph of order n.

- Is there a constant $c > 0$ such that $\delta_4(n) \geq cn$?
- What is the order of magnitude of $\delta_5(n)$?
- Do r-regular critical 4-chromatic graphs exist for all $r \geq 3$?

P. Erdős 1949,1989

Minimal degrees and regularity.

Let $\delta_k(n)$ be the largest minimal degree of a critical k-chromatic graph of order n.

- Is there a constant $c > 0$ such that $\delta_4(n) \geq cn$?
- What is the order of magnitude of $\delta_5(n)$?
- Do r-regular critical 4-chromatic graphs exist for all $r \geq 3$?

P. Erdős 1949,1989

Minimal degrees and regularity.

Let $\delta_k(n)$ be the largest minimal degree of a critical k-chromatic graph of order n.

- Is there a constant $c > 0$ such that $\delta_4(n) \geq cn$?
- What is the order of magnitude of $\delta_5(n)$?
- Do r-regular critical 4-chromatic graphs exist for all $r \geq 3$?

P. Erdős 1949,1989

Minimal degrees and regularity.

Let $\delta_k(n)$ be the largest minimal degree of a critical k-chromatic graph of order n.

- Is there a constant $c > 0$ such that $\delta_4(n) \geq cn$?
- What is the order of magnitude of $\delta_5(n)$?
- Do r-regular critical 4-chromatic graphs exist for all $r \geq 3$?

P. Erdős 1949,1989

Minimal degrees and regularity.

Let $\delta_k(n)$ be the largest minimal degree of a critical k-chromatic graph of order n.

- Is there a constant $c > 0$ such that $\delta_4(n) \geq cn$?
- What is the order of magnitude of $\delta_5(n)$?
- Do r-regular critical 4-chromatic graphs exist for all $r \geq 3$?

P. Erdős 1949,1989

Minimal degrees and regularity.

- $\delta_4(n) \geq c\sqrt[3]{n}$.

  M. Simonovits 1972 & B. Toft 1972

- $\delta_6(n) \geq n/2$.

  G.A. Dirac 1949

- There are infinitely many 4-regular critical 4-chromatic graphs.

  T. Gallai 1963

- There are infinitely many 5-regular critical 4-chromatic graphs.

  T.R.J. 2002

- There exists a r-regular critical 4-chromatic graph for each $r = 6, 8, 10$.

  A.A. Dobrynin, L.S. Mel'nikov & A.V. Pyatkin 2003

Minimal degrees and regularity.

- $\delta_4(n) \geq c\sqrt[3]{n}$.

  M. Simonovits 1972 & B. Toft 1972

- $\delta_6(n) \geq n/2$.

  G.A. Dirac 1949

- There are infinitely many 4-regular critical 4-chromatic graphs.

  T. Gallai 1963

- There are infinitely many 5-regular critical 4-chromatic graphs.

  T.R.J. 2002

- There exists a r-regular critical 4-chromatic graph for each $r = 6, 8, 10$.

  A.A. Dobrynin, L.S. Mel'nikov & A.V. Pyatkin 2003

Minimal degrees and regularity.

- $\delta_4(n) \geq c\sqrt[3]{n}$.

  M. Simonovits 1972 & B. Toft 1972

- $\delta_6(n) \geq n/2$.

  G.A. Dirac 1949

- There are infinitely many 4-regular critical 4-chromatic graphs.

  T. Gallai 1963

- There are infinitely many 5-regular critical 4-chromatic graphs.

  T.R.J. 2002

- There exists a r-regular critical 4-chromatic graph for each $r = 6, 8, 10$.

  A.A. Dobrynin, L.S. Mel'nikov & A.V. Pyatkin 2003

Minimal degrees and regularity.

- $\delta_4(n) \geq c\sqrt[3]{n}$.

  M. Simonovits 1972 & B. Toft 1972
- $\delta_6(n) \geq n/2$.

  G.A. Dirac 1949
- There are infinitely many 4-regular critical 4-chromatic graphs.

  T. Gallai 1963
- There are infinitely many 5-regular critical 4-chromatic graphs.

  T.R.J. 2002
- There exists a r-regular critical 4-chromatic graph for each $r = 6, 8, 10$.

  A.A. Dobrynin, L.S. Mel'nikov & A.V. Pyatkin 2003

Minimal degrees and regularity.

- $\delta_4(n) \geq c\sqrt[3]{n}$.

  M. Simonovits 1972 & B. Toft 1972
- $\delta_6(n) \geq n/2$.

  G.A. Dirac 1949
- There are infinitely many 4-regular critical 4-chromatic graphs.

  T. Gallai 1963
- There are infinitely many 5-regular critical 4-chromatic graphs.

  T.R.J. 2002
- There exists a r-regular critical 4-chromatic graph for each $r = 6, 8, 10$.

  A.A. Dobrynin, L.S. Mel'nikov & A.V. Pyatkin 2003

Non-amenable critical graphs.

G is amenable if for every non-constant $f : V \to \{1, 2, \ldots, k\}$ there exists a k-coloring $\varphi : V \to \{1, 2, \ldots, k\}$ such that $\varphi(v) \neq f(v)$ for all $v \in V$.

- What is the minimal order n_k of a critical k-chromatic graph which is not amenable?

T. Gallai 1963
B. Toft 1970

Non-amenable critical graphs.

G is amenable if for every non-constant $f : V \to \{1, 2, \ldots, k\}$ there exists a k-coloring $\varphi : V \to \{1, 2, \ldots, k\}$ such that $\varphi(v) \neq f(v)$ for all $v \in V$.

- What is the minimal order n_k of a critical k-chromatic graph which is not amenable?

T. Gallai 1963
B. Toft 1970

Non-amenable critical graphs.

G is amenable if for every non-constant $f : V \to \{1, 2, \ldots, k\}$ there exists a k-coloring $\varphi : V \to \{1, 2, \ldots, k\}$ such that $\varphi(v) \neq f(v)$ for all $v \in V$.

- What is the minimal order n_k of a critical k-chromatic graph which is not amenable?

T. Gallai 1963
B. Toft 1970

Non-amenable critical graphs.

- $n_k \leq 11k - 24$ for all $k \geq 5$.

  B.Aa. Sørensen & B. Toft 1974
- $n_4 \leq 50$.

  B. Toft 1987
- $n_4 \geq 10$.

  J.I. Brown, D. Kelly, J. Schönheim & R.E. Woodrow 1990

Non-amenable critical graphs.

- $n_k \leq 11k - 24$ for all $k \geq 5$.

  B.Aa. Sørensen & B. Toft 1974
- $n_4 \leq 50$.

  B. Toft 1987
- $n_4 \geq 10$.

  J.I. Brown, D. Kelly, J. Schönheim & R.E. Woodrow 1990

Non-amenable critical graphs.

- $n_k \leq 11k - 24$ for all $k \geq 5$.
  B.Aa. Sørensen & B. Toft 1974
- $n_4 \leq 50$.
  B. Toft 1987
- $n_4 \geq 10$.
  J.I. Brown, D. Kelly, J. Schönheim & R.E. Woodrow 1990

Degree 5 problem.

- Assuming that
  - G is critical 4-chromatic,
  - v is a vertex of degree 5 in G and $N(v)$ is its set of neighbors,
  - $f : N(v) \to \{1, 2, 3\}$ is a non-constant map,

  does a 3-coloring $\varphi : V \setminus \{v\} \to \{1, 2, 3\}$ of $G - v$ exist that satisfies $\varphi(x) \neq f(x)$ for all $x \in N(v)$?

B. Toft 1974

Degree 5 problem.

- Assuming that
  - G is critical 4-chromatic,
  - v is a vertex of degree 5 in G and $N(v)$ is its set of neighbors,
  - $f : N(v) \to \{1,2,3\}$ is a non-constant map,

  does a 3-coloring $\varphi : V \setminus \{v\} \to \{1,2,3\}$ of $G - v$ exist that satisfies $\varphi(x) \neq f(x)$ for all $x \in N(v)$?

B. Toft 1974

Degree 5 problem.

- Assuming that
  - G is critical 4-chromatic,
  - v is a vertex of degree 5 in G and $N(v)$ is its set of neighbors,
  - $f : N(v) \to \{1, 2, 3\}$ is a non-constant map,

  does a 3-coloring $\varphi : V \setminus \{v\} \to \{1, 2, 3\}$ of $G - v$ exist that satisfies $\varphi(x) \neq f(x)$ for all $x \in N(v)$?

B. Toft 1974

Degree 5 problem.

- Assuming that
  - G is critical 4-chromatic,
  - v is a vertex of degree 5 in G and $N(v)$ is its set of neighbors,
  - $f : N(v) \to \{1, 2, 3\}$ is a non-constant map,

does a 3-coloring $\varphi : V \setminus \{v\} \to \{1, 2, 3\}$ of $G - v$ exist that satisfies $\varphi(x) \neq f(x)$ for all $x \in N(v)$?

B. Toft 1974

Degree 5 problem.

- Assuming that
  - G is critical 4-chromatic,
  - v is a vertex of degree 5 in G and $N(v)$ is its set of neighbors,
  - $f : N(v) \to \{1, 2, 3\}$ is a non-constant map,

  does a 3-coloring $\varphi : V \setminus \{v\} \to \{1, 2, 3\}$ of $G - v$ exist that satisfies $\varphi(x) \neq f(x)$ for all $x \in N(v)$?

B. Toft 1974

Degree 5 problem.

- True for any vertex v of degree < 5, but false for vertices of degree > 5.
- For critical k-chromatic graphs, and precoloring with $k-1$ colors, when $k > 4$: true for vertices of degree $\leq 2k-3$, and false for degree $\geq 2k-2$.

B. Toft 1974

Degree 5 problem.

- True for any vertex v of degree < 5, but false for vertices of degree > 5.
- For critical k-chromatic graphs, and precoloring with $k-1$ colors, when $k > 4$: true for vertices of degree $\leq 2k-3$, and false for degree $\geq 2k-2$.

B. Toft 1974

Fixed subgraphs.

Assume that a graph F is a subgraph of some critical k-chromatic graph.

- Is F a subgraph of a critical k-chromatic graph H of order

$$|V(H)| \leq c_k |V(F)|,$$

where c_k depends only on k?

M. Stiebitz 1987

Fixed subgraphs.

Assume that a graph F is a subgraph of some critical k-chromatic graph.

- Is F a subgraph of a critical k-chromatic graph H of order

$$|V(H)| \leq c_k|V(F)|,$$

where c_k depends only on k?

M. Stiebitz 1987

Fixed subgraphs.

Assume that a graph F is a subgraph of some critical k-chromatic graph.

- Is F a subgraph of a critical k-chromatic graph H of order

$$|V(H)| \leq c_k |V(F)|,$$

where c_k depends only on k?

M. Stiebitz 1987

Fixed subgraphs.

- If F is any $(k-2)$-colorable graph, there exists a critical k-chromatic graph H with F as subgraph, and satisfying

$$|V(H)| \leq 2|V(F)| + d_k,$$

where d_k depends only on k.

B. Toft 1974

Smallest circumference.

- What is the minimal value $L_k(n)$ of the length of a longest cycle in a critical k-chromatic graph of order at least n?

J.B. Kelly and L.M. Kelly 1954

Smallest circumference.

- What is the minimal value $L_k(n)$ of the length of a longest cycle in a critical k-chromatic graph of order at least n?

J.B. Kelly and L.M. Kelly 1954

Smallest circumference.

Bounds for $k \geq 4$:

- $L_k(n) < 2(k-1)\log n/\log(k-2)$ for infinitely many n.
  T. Gallai 1963
- $L_k(n) \geq 2\sqrt{\log(n-1)/\log(k-2)}$.
  N. Alon, M. Krivelevich & P.D. Seymour 2000

Smallest circumference.

Bounds for $k \geq 4$:

- $L_k(n) < 2(k-1)\log n/\log(k-2)$ for infinitely many n.
  T. Gallai 1963
- $L_k(n) \geq 2\sqrt{\log(n-1)/\log(k-2)}$.
  N. Alon, M. Krivelevich & P.D. Seymour 2000

Smallest circumference.

Bounds for $k \geq 4$:

- $L_k(n) < 2(k-1)\log n/\log(k-2)$ for infinitely many n.
  T. Gallai 1963
- $L_k(n) \geq 2\sqrt{\log(n-1)/\log(k-2)}$.
  N. Alon, M. Krivelevich & P.D. Seymour 2000

Erdős & Lovász Tihany Problem.

Assume $G \neq K_k$. Choose $a, b \geq 2$ so that $a + b = k + 1$.

- Does G contain two disjoint subgraphs of chromatic numbers a and b?
- In particular ($b = 2$), is K_k the only k-chromatic double-critical graph? That is, if $G - u - v$ is $(k-2)$-colorable for every edge $uv \in E$, then this implies $G = K_k$?

P. Erdős and L. Lovász 1968

Erdős & Lovász Tihany Problem.

Assume $G \neq K_k$. Choose $a, b \geq 2$ so that $a + b = k + 1$.

- Does G contain two disjoint subgraphs of chromatic numbers a and b?
- In particular ($b = 2$), is K_k the only k-chromatic double-critical graph? That is, if $G - u - v$ is $(k-2)$-colorable for every edge $uv \in E$, then this implies $G = K_k$?

P. Erdős and L. Lovász 1968

Erdős & Lovász Tihany Problem.

Assume $G \neq K_k$. Choose $a, b \geq 2$ so that $a + b = k + 1$.

- Does G contain two disjoint subgraphs of chromatic numbers a and b?
- In particular ($b = 2$), is K_k the only k-chromatic double-critical graph? That is, if $G - u - v$ is $(k-2)$-colorable for every edge $uv \in E$, then this implies $G = K_k$?

P. Erdős and L. Lovász 1968

Erdős & Lovász Tihany Problem.

Assume $G \neq K_k$. Choose $a, b \geq 2$ so that $a + b = k + 1$.

- Does G contain two disjoint subgraphs of chromatic numbers a and b?
- In particular ($b = 2$), is K_k the only k-chromatic double-critical graph? That is, if $G - u - v$ is $(k - 2)$-colorable for every edge $uv \in E$, then this implies $G = K_k$?

P. Erdős and L. Lovász 1968

Erdős & Lovász Tihany Problem.

- True for $(k, a, b) \in \{(4, 2, 3), (5, 2, 4), (5, 3, 3), (6, 3, 4), (7, 3, 5)\}$.

W.G. Brown & H.A. Jung 1969
N.N. Mozhan 1986
M. Stiebitz 1987, 1988

Large critical subgraphs.

- Does every critical k-chromatic graph contain a large critical $(k-1)$-chromatic subgraph?

J. Nešetřil and V. Rödl 1973

Large critical subgraphs.

- Does every critical k-chromatic graph contain a large critical $(k-1)$-chromatic subgraph?

J. Nešetřil and V. Rödl 1973

Large critical subgraphs.

- True for $k = 4$.

  J.B. Kelly & L.M. Kelly 1954
  H.-J. Voss 1977, 1991

- If every critical $(k-1)$-chromatic subgraph is a K_{k-1}, then G is a K_k.

  M. Stiebitz 1987

Large critical subgraphs.

- True for $k = 4$.

  J.B. Kelly & L.M. Kelly 1954
  H.-J. Voss 1977, 1991

- If every critical $(k-1)$-chromatic subgraph is a K_{k-1}, then G is a K_k.

  M. Stiebitz 1987

Critical subgraph with a given path.

Let P be a path of length 2 in G.

- Is there a critical $(k-1)$-chromatic subgraph of G which contains P?

B. Toft 1974

Critical subgraph with a given path.

Let P be a path of length 2 in G.

- Is there a critical $(k-1)$-chromatic subgraph of G which contains P?

B. Toft 1974

Critical subgraph with a given path.

Let P be a path of length 2 in G.

- Is there a critical $(k-1)$-chromatic subgraph of G which contains P?

B. Toft 1974

Critical subgraph with a given path.

- True for $k = 4$.

G.A. Dirac 1964

- For all $e_1, e_2 \in E$ there exists a critical $(k-1)$-chromatic subgraph of G containing e_1 but not e_2.

B. Toft 1974

Critical subgraph with a given path.

- True for $k = 4$.

G.A. Dirac 1964

- For all $e_1, e_2 \in E$ there exists a critical $(k-1)$-chromatic subgraph of G containing e_1 but not e_2.

B. Toft 1974

Partial joins.

G is a partial join of G_1 and G_2 if both are induced subgraphs of G, and if $V = V(G_1) \cup V(G_2)$.

- Characterize the critical graphs that are partial joins of other critical graphs.
- In particular, when is a partial join of two odd cycles a critical graph?
- When is a partial join of a complete graph with another critical graph again critical?

T. Gallai 1969

Partial joins.

G is a partial join of G_1 and G_2 if both are induced subgraphs of G, and if $V = V(G_1) \cup V(G_2)$.

- Characterize the critical graphs that are partial joins of other critical graphs.
- In particular, when is a partial join of two odd cycles a critical graph?
- When is a partial join of a complete graph with another critical graph again critical?

T. Gallai 1969

Partial joins.

G is a partial join of G_1 and G_2 if both are induced subgraphs of G, and if $V = V(G_1) \cup V(G_2)$.

- Characterize the critical graphs that are partial joins of other critical graphs.
- In particular, when is a partial join of two odd cycles a critical graph?
- When is a partial join of a complete graph with another critical graph again critical?

T. Gallai 1969

Partial joins.

G is a partial join of G_1 and G_2 if both are induced subgraphs of G, and if $V = V(G_1) \cup V(G_2)$.

- Characterize the critical graphs that are partial joins of other critical graphs.
- In particular, when is a partial join of two odd cycles a critical graph?
- When is a partial join of a complete graph with another critical graph again critical?

T. Gallai 1969

Partial joins.

G is a partial join of G_1 and G_2 if both are induced subgraphs of G, and if $V = V(G_1) \cup V(G_2)$.

- Characterize the critical graphs that are partial joins of other critical graphs.
- In particular, when is a partial join of two odd cycles a critical graph?
- When is a partial join of a complete graph with another critical graph again critical?

T. Gallai 1969

Partial joins.

- If a partial join of two complete graphs is critical, then the join is a complete join ($\Leftrightarrow$ the complement of a bipartite graph is perfect).

  T. Gallai 1969

- A characterization of critical partial joins of complete graphs with odd cycles is known.

  M. Stiebitz & W. Wessel 1993

Partial joins.

- If a partial join of two complete graphs is critical, then the join is a complete join ($\Leftrightarrow$ the complement of a bipartite graph is perfect).

  T. Gallai 1969

- A characterization of critical partial joins of complete graphs with odd cycles is known.

  M. Stiebitz & W. Wessel 1993

Number of critical subgraphs.

- Does a critical k-chromatic graph of order n contain n distinct critical $(k-1)$-chromatic subgraphs?
- Does a vertex-critical k-chromatic graph of order n contain n distinct vertex-critical $(k-1)$-chromatic subgraphs?
- If $G \neq K_k$, is there an edge of G that is contained in at most one complete subgraph of order $k-1$?

T. Gallai 1984
Xiang-Ying Su 1994

Number of critical subgraphs.

- Does a critical k-chromatic graph of order n contain n distinct critical $(k-1)$-chromatic subgraphs?
- Does a vertex-critical k-chromatic graph of order n contain n distinct vertex-critical $(k-1)$-chromatic subgraphs?
- If $G \neq K_k$, is there an edge of G that is contained in at most one complete subgraph of order $k-1$?

T. Gallai 1984
Xiang-Ying Su 1994

Number of critical subgraphs.

- Does a critical k-chromatic graph of order n contain n distinct critical $(k-1)$-chromatic subgraphs?
- Does a vertex-critical k-chromatic graph of order n contain n distinct vertex-critical $(k-1)$-chromatic subgraphs?
- If $G \neq K_k$, is there an edge of G that is contained in at most one complete subgraph of order $k-1$?

T. Gallai 1984
Xiang-Ying Su 1994

Number of critical subgraphs.

- Does a critical k-chromatic graph of order n contain n distinct critical $(k-1)$-chromatic subgraphs?
- Does a vertex-critical k-chromatic graph of order n contain n distinct vertex-critical $(k-1)$-chromatic subgraphs?
- If $G \neq K_k$, is there an edge of G that is contained in at most one complete subgraph of order $k-1$?

T. Gallai 1984
Xiang-Ying Su 1994

Number of critical subgraphs.

- A critical k-chromatic graph contains at least $\log n$ distinct critical $(k-1)$-chromatic subgraphs.

  M. Stiebitz 1985
- A vertex-critical k-chromatic graph contains at least $\sqrt[k-1]{n(k-1)!}$ vertex-critical $(k-1)$-chromatic subgraphs.

  H.L. Abbott & Bing Zhou 1992
- If $k \leq 7$, and if $G \neq K_k$, then there is an edge of G that is contained in at most one complete subgraph of order $k-1$.

  Xiang-Ying Su 1994

Number of critical subgraphs.

- A critical k-chromatic graph contains at least $\log n$ distinct critical $(k-1)$-chromatic subgraphs.

  M. Stiebitz 1985

- A vertex-critical k-chromatic graph contains at least $\sqrt[k-1]{n(k-1)!}$ vertex-critical $(k-1)$-chromatic subgraphs.

  H.L. Abbott & Bing Zhou 1992

- If $k \leq 7$, and if $G \neq K_k$, then there is an edge of G that is contained in at most one complete subgraph of order $k-1$.

  Xiang-Ying Su 1994

Number of critical subgraphs.

- A critical k-chromatic graph contains at least $\log n$ distinct critical $(k-1)$-chromatic subgraphs.

  M. Stiebitz 1985
- A vertex-critical k-chromatic graph contains at least $\sqrt[k-1]{n(k-1)!}$ vertex-critical $(k-1)$-chromatic subgraphs.

  H.L. Abbott & Bing Zhou 1992
- If $k \leq 7$, and if $G \neq K_k$, then there is an edge of G that is contained in at most one complete subgraph of order $k-1$.

  Xiang-Ying Su 1994

Vertex-critical graphs without critical edges.

An element $x \in V \cup E$ is critical if $\chi(G - x) < \chi(G)$.

- Does there exist a vertex-critical 4-chromatic graph without critical edges?
- Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that there exists for each $k \geq 5$ a vertex-critical k-chromatic graph G_k with

$$\chi(G_k - A) = k \quad \text{for all} \quad A \subset E(G_k) \quad \text{with} \quad |A| \leq f(|V(G_k)|)?$$

If f exists, how fast may it increase?

G.A. Dirac ?
P. Erdős 1989

Vertex-critical graphs without critical edges.

An element $x \in V \cup E$ is critical if $\chi(G - x) < \chi(G)$.

- Does there exist a vertex-critical 4-chromatic graph without critical edges?
- Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that there exists for each $k \geq 5$ a vertex-critical k-chromatic graph G_k with

$$\chi(G_k - A) = k \quad \text{for all} \quad A \subset E(G_k) \quad \text{with} \quad |A| \leq f(|V(G_k)|)?$$

If f exists, how fast may it increase?

G.A. Dirac ?
P. Erdős 1989

Vertex-critical graphs without critical edges.

An element $x \in V \cup E$ is critical if $\chi(G - x) < \chi(G)$.

- Does there exist a vertex-critical 4-chromatic graph without critical edges?
- Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that there exists for each $k \geq 5$ a vertex-critical k-chromatic graph G_k with

  $$\chi(G_k - A) = k \text{ for all } A \subset E(G_k) \text{ with } |A| \leq f(|V(G_k)|)?$$

  If f exists, how fast may it increase?

G.A. Dirac ?
P. Erdős 1989

Vertex-critical graphs without critical edges.

An element $x \in V \cup E$ is critical if $\chi(G - x) < \chi(G)$.

- Does there exist a vertex-critical 4-chromatic graph without critical edges?
- Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that there exists for each $k \geq 5$ a vertex-critical k-chromatic graph G_k with

$$\chi(G_k - A) = k \text{ for all } A \subset E(G_k) \text{ with } |A| \leq f(|V(G_k)|)?$$

If f exists, how fast may it increase?

G.A. Dirac ?
P. Erdős 1989

Vertex-critical graphs without critical edges.

- For each $k \geq 5$ there are infinitely many vertex-critical k-chromatic graphs without critical edges. Still the bound $f \geq 1$ is the best known lower bound on the growth of f.
- There exists a vertex-critical 5-chromatic graph with two edge-disjoint critical 5-chromatic subgraphs.
- There exists a vertex-critical 5-chromatic graph H satisfying

$$\chi(H - A) = k \quad \text{for all} \quad A \subset E(H) \quad \text{with} \quad |A| \leq 2$$

T.R.J. 2002

Vertex-critical graphs without critical edges.

- For each $k \geq 5$ there are infinitely many vertex-critical k-chromatic graphs without critical edges. Still the bound $f \geq 1$ is the best known lower bound on the growth of f.
- There exists a vertex-critical 5-chromatic graph with two edge-disjoint critical 5-chromatic subgraphs.
- There exists a vertex-critical 5-chromatic graph H satisfying

$$\chi(H - A) = k \quad \text{for all} \quad A \subset E(H) \quad \text{with} \quad |A| \leq 2$$

T.R.J. 2002

Vertex-critical graphs without critical edges.

- For each $k \geq 5$ there are infinitely many vertex-critical k-chromatic graphs without critical edges. Still the bound $f \geq 1$ is the best known lower bound on the growth of f.
- There exists a vertex-critical 5-chromatic graph with two edge-disjoint critical 5-chromatic subgraphs.
- There exists a vertex-critical 5-chromatic graph H satisfying

$$\chi(H - A) = k \text{ for all } A \subset E(H) \text{ with } |A| \leq 2$$

T.R.J. 2002

Construction of critical graphs.

- Does a construction of critical k-chromatic graphs exist, starting from K_k and applying elementary steps in which each intermediate graph is itself critical k-chromatic?

G. Hajós 1961

Construction of critical graphs.

- Does a construction of critical k-chromatic graphs exist, starting from K_k and applying elementary steps in which each intermediate graph is itself critical k-chromatic?

G. Hajós 1961

Construction of critical graphs.

- The classical Hajós and Ore constructions do not achieve this for any value $k \geq 4$.

D. Hanson, G.C. Robinson & B. Toft 1986

T.R.J. & G.F. Royle 1999

Summary

- Critical graphs have been studied for the past 60 years by a number of the most prominent combinatorialists.
- And yet, almost all classical problems on critical graphs remain open.

- Outlook
  - Complexity issues for hard problems gain increasing importance. Perhaps this may lead to renewed interest in problems about criticality.

Summary

- Critical graphs have been studied for the past 60 years by a number of the most prominent combinatorialists.
- And yet, almost all classical problems on critical graphs remain open.
- Outlook
  - Complexity issues for hard problems gain increasing importance. Perhaps this may lead to renewed interest in problems about criticality.

Summary

- Critical graphs have been studied for the past 60 years by a number of the most prominent combinatorialists.
- And yet, almost all classical problems on critical graphs remain open.

- Outlook
  - Complexity issues for hard problems gain increasing importance. Perhaps this may lead to renewed interest in problems about criticality.

Summary

- Critical graphs have been studied for the past 60 years by a number of the most prominent combinatorialists.
- And yet, almost all classical problems on critical graphs remain open.

- Outlook
  - Complexity issues for hard problems gain increasing importance. Perhaps this may lead to renewed interest in problems about criticality.

Thank you!

For Further Reading I

T.R. Jensen & B. Toft
Graph Coloring Problems (Chapter 5).
Wiley-Interscience 1995.

http://www.imada.sdu.dk/∼btoft/graphcol