A Dice Rolling Game on a Set of Toruses

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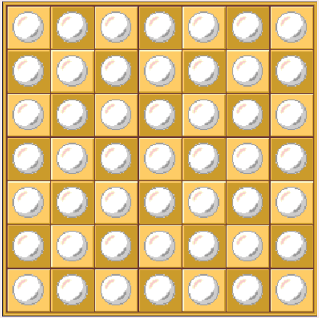
Fiver game

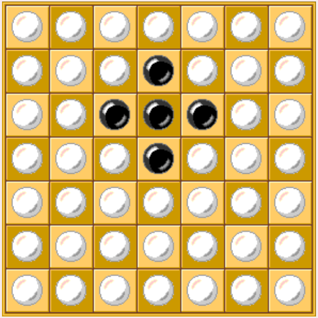
Fiver Game



7×7 board

Fiver game

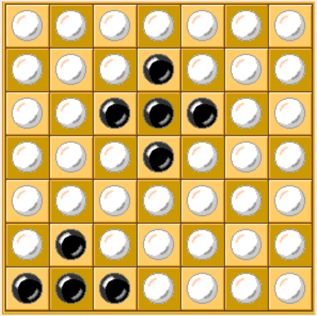
Fiver Game



7×7 board

Fiver game

Fiver Game



7×7 board

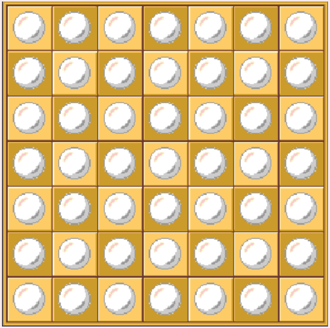
Fiver game

Fiver Game

"Is it possible to change from being completely full of white pieces to entirely black pieces?"

A variant of Fiver game

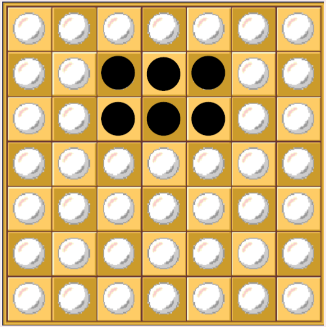
A variant of Fiver game



7×7 board

A variant of Fiver game

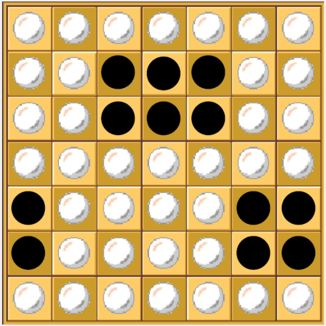
A variant of Fiver game



7×7 board

A variant of Fiver game

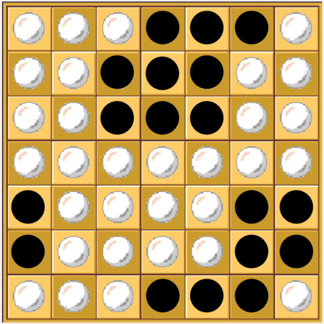
A variant of Fiver game



7 × 7 board

A variant of Fiver game

A variant of Fiver game



7 × 7 board

A variant of Fiver game

2	0	0	1	0	0	2
0	1	1	2	1	1	0
1	0	1	1	0	1	0
2	0	1	0	1	0	0
2	0	0	1	2	0	2
0	0	0	2	0	1	0
0	2	2	1	2	0	1

2	0	0	1	1	1	0
0	1	1	2	1	1	0
1	0	1	1	0	1	0
2	0	1	0	1	0	0
2	0	0	1	2	0	2
0	0	0	2	0	1	0
0	2	2	1	0	1	2

2	0	0	1	1	1	0
0	1	1	2	1	1	0
1	0	1	1	0	1	0
0	0	1	0	1	1	1
0	0	0	1	2	1	0
0	0	0	2	0	1	0
0	2	2	1	0	1	2

Elements of Z_3 are located on 7×7 board

A variant of Fiver game

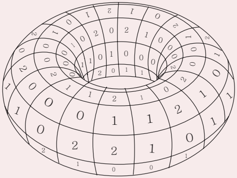
A variant of Fiver game

"Given a torus, is it possible to have 0 appear on the top face of each square on the torus?"

A Dice Rolling Game on a set of Toruses

A Dice Rolling Game on a set of Toruses

- Given a positive integer n, an *n-dice* is a dice with n faces such that element i of $\mathbb{Z}_n$ is written on the ith face.
- Given positive integers α_1, α_2, we arbitrarily locate $\alpha_1\alpha_2$ n-dice in an $\alpha_1 \times \alpha_2$ rectangular array, and glue the lower and upper together and also the left and right edges.
  $\Rightarrow$ $\alpha_1\alpha_2$ n-dice on a torus



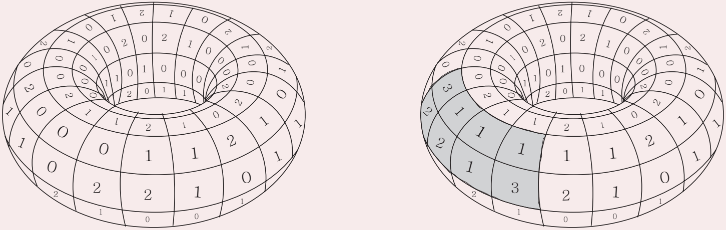
A Dice Rolling Game on a set of Toruses

A Dice Rolling Game on a set of Toruses

- $\mathcal{D}((\alpha_1, \alpha_2), n)$: the set of toruses on each of which $\alpha_1\alpha_2$ n-dice are located.
- "(β_1, β_2)-rolling procedure": For positive integers β_1, β_2, $\beta_1 \leq \alpha_1$, $\beta_2 \leq \alpha_2$ and a torus belonging to $\mathcal{D}((\alpha_1, \alpha_2), n)$, we roll the dice which form a $\beta_1 \times \beta_2$ rectangular array on the torus so that we increase the number on the top face of each of them by 1.

A Dice Rolling Game on a set of Toruses

Example: "(β_1, β_2)-rolling procedure"



A torus in $\mathcal{D}((19, 8), 3)$ and the torus resulting from going through a $(2, 4)$-rolling procedure

A Dice Rolling Game on a set of Toruses

Definition of $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game

"Given a torus in $\mathcal{D}((\alpha_1, \alpha_2), n)$, is it possible to have 0 appear on the top face of each of $\alpha_1\alpha_2$ n-dice on the torus by repeatedly applying (β_1, β_2)-rolling procedures?"

We call this game the *dice rolling game on $\mathcal{D}((\alpha_1, \alpha_2), n)$ with respect to (β_1, β_2)-rolling procedures* or the $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$*-DR game* for short.

solution of the $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game

We say that a torus for which the answer to the above question is yes is a *solution of the $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game*.

$((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game

Basic Notations

- $\mathcal{M}((\alpha_1, \alpha_2), n)$: the set of $\alpha_1 \times \alpha_2$ matrices with elements in $\mathbb{Z}_n$
- For each element $A \in \mathcal{M}((\alpha_1, \alpha_2), n)$, we denote by $[A]_{i,j}$ the element of $\mathbb{Z}_n$ in the (i,j)-entry.
- Given A, $B \in \mathcal{M}((\alpha_1, \alpha_2), n)$,

$$[A+B]_{i,j} = [A]_{i,j} + [B]_{i,j} \quad \text{and} \quad [cA]_{i,j} = c[A]_{i,j}$$

  for any $c \in R$, $0 \le i \le \alpha_1 - 1$, $0 \le j \le \alpha_2 - 1$.
- $E_{i,j}$: $\alpha_1 \times \alpha_2$ matrix with 1 in the (i,j)-entry and 0 elsewhere.

$((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game

Basic Notations

- For positive integers m_1, m_2 and integers k_1, k_2 $0 \le k_1 \le \alpha_1 - 1$, $0 \le k_2 \le \alpha_2 - 1$,

$$J_{k_1,k_2}^{(m_1,m_2)} = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} E_{k_1+i,k_2+j}$$

(example) $\alpha_1 = 6$, $\alpha_2 = 8$, $\beta_1 = 2$, $\beta_2 = 2$, $n = 5$.

$J_{1,2}^{(3,2)} =$

0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

.

- Especially, we denote $J_{k_1,k_2}^{(\beta_1,\beta_2)}$ by J_{k_1,k_2}^{*}.

$((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game

Solution Matrix

- A is a *solution matrix of the* $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$*-DR game* if A can be O by adding a linear combination of $J^*_{\cdot,\cdot}$.
- A is a solution matrix of the $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game if and only if there exist $c_{i,j} \in \mathbb{Z}$ such that $A + \sum_{j=0}^{\alpha_2 - 1} \sum_{i=0}^{\alpha_1 - 2} c_{i,j} J^*_{i,j} = O$. We call matrix $(c_{i,j})$ a *solving coefficient matrix* of the $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game corresponding to A.

Main Results:

Characterizing solution matrices of the $((\alpha_1, \alpha_2); (\beta_1, \beta_2); n)$-DR game

Definitions for $C_{i,j}$, $R_{i,j}$ and $Q_{i,j}$

- $g_i = \gcd(\alpha_i, \beta_i)$ for $i = 1, 2$ and $\beta_i = s_i g_i$.
- For integers i, j,
  $\exists$ integers t_i, u_i, v_j, w_j $(0 \le u_i \le g_1 - 1,\ 0 \le w_j \le g_2 - 1)$
  s.t. $i = t_i g_1 + u_i$, $j = v_j g_2 + w_j$.
  $\exists$ positive integers ζ_i, η_j
  s.t. $t_i g_1 \equiv \zeta_i \beta_1 \pmod{\alpha_1}$, $v_j g_2 \equiv \eta_j \beta_2 \pmod{\alpha_2}$.

For integers i, j, $(0 \le i \le \alpha_1 - 1, 0 \le j \le \alpha_2 - 1)$

$$C_{i,j} = \sum_{m=0}^{\eta_j - 1} \left(J^*_{i, w_j + m\beta_2} - J^*_{i, w_j + m\beta_2 + 1} \right)$$

$$R_{i,j} = \sum_{m=0}^{\zeta_i - 1} \left(J^*_{u_i + m\beta_1, j} - J^*_{u_i + m\beta_1 + 1, j} \right)$$

$$Q_{i,j} = \sum_{m=0}^{\zeta_i - 1} \left(C_{u_i + m\beta_1, j} - C_{u_i + m\beta_1 + 1, j} \right)$$

$C_{i,j}$

Example

In the $((6,8);(2,4);5)$-DR game, $g_1 = 2$ and $g_2 = 4$,

$C_{2,7} =$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	−1
0	0	0	1	0	0	0	−1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

.

$C_{2,7} = J^*_{2,3} - J^*_{2,4}$

$=$

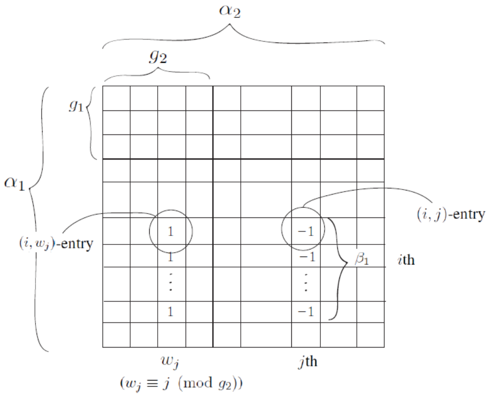
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0
0	0	0	1	1	1	1	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$-$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1
0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$C_{i,j} = \sum_{a=0}^{\beta_1 - 1} (E_{i+a,w_j} - E_{i+a,j})$

$g_i = \gcd(\alpha_i, \beta_i)$ for $i = 1, 2$ and $\beta_i = s_i g_i$.



$C_{i,j} = O$ if $0 \leq j \leq g_2 - 1$

$C_{i,j} = \sum_{a=0}^{\beta_1-1}(E_{i+a,w_j} - E_{i+a,j})$

$g_i = \gcd(\alpha_i, \beta_i)$ for $i = 1, 2$ and $\beta_i = s_i g_i$.

$$
\begin{aligned}
C_{i,j} &= \sum_{m=0}^{\eta_j-1}\left(J^*_{i,w_j+m\beta_2} - J^*_{i,w_j+m\beta_2+1}\right) \\
&= \sum_{m=0}^{\eta_j-1}\left[\left(\sum_{a=0}^{\beta_1-1}\sum_{b=0}^{\beta_2-1} E_{i+a,w_j+m\beta_2+b}\right) - \left(\sum_{a=0}^{\beta_1-1}\sum_{b=0}^{\beta_2-1} E_{i+a,w_j+m\beta_2+b+1}\right)\right] \\
&= \sum_{m=0}^{\eta_j-1}\sum_{a=0}^{\beta_1-1}\sum_{b=0}^{\beta_2-1}\left(E_{i+a,w_j+m\beta_2+b} - E_{i+a,w_j+m\beta_2+b+1}\right) \\
&= \sum_{m=0}^{\eta_j-1}\sum_{a=0}^{\beta_1-1}(E_{i+a,w_j+m\beta_2} - E_{i+a,w_j+(m+1)\beta_2}) \\
&= \sum_{a=0}^{\beta_1-1}(E_{i+a,w_j} - E_{i+a,w_j+\eta_j\beta_2}) = \sum_{a=0}^{\beta_1-1}(E_{i+a,w_j} - E_{i+a,j}).
\end{aligned}
$$

$R_{i,j}$

Example

In the $((6,8);(2,4);5)$-DR game, $g_1 = 2$ and $g_2 = 4$,

$R_{5,4} =$

0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	−1	−1	−1	−1

.

$$R_{5,4} = (J^*_{1,4} - J^*_{2,4}) + (J^*_{3,4} - J^*_{4,4})$$

=

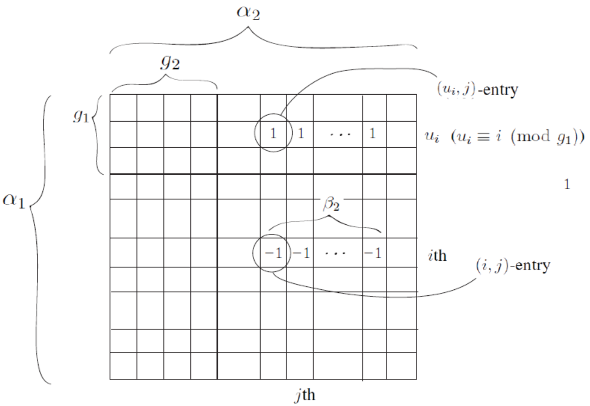
0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	−1	−1	−1	−1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

−

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	−1	−1	−1	−1

$R_{i,j} = \sum_{b=0}^{\beta_2-1}(E_{u_i,j+b} - E_{i,j+b})$

$g_i = \gcd(\alpha_i, \beta_i)$ for $i = 1, 2$ and $\beta_i = s_i g_i$.



$R_{i,j} = O$ if $0 \le i \le g_1 - 1$

$Q_{i,j}$

Example

In the $((6,8);(2,4);5)$-DR game, $g_1 = 2$ and $g_2 = 4$,

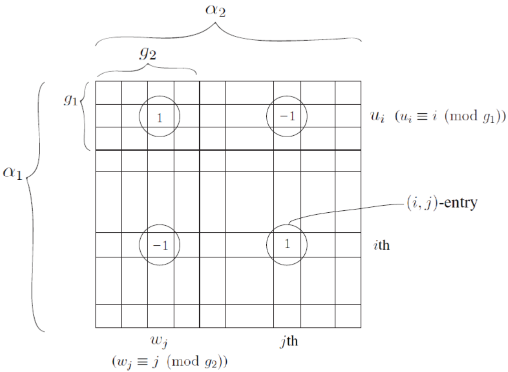
$Q_{3,7}$ =

0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	−1
0	0	0	0	0	0	0	0
0	−1	0	0	0	0	0	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

.

$Q_{i,j} = E_{u_i,w_j} - E_{i,w_j} - E_{u_i,j} + E_{i,j}$

$g_i = \gcd(\alpha_i, \beta_i)$ for $i = 1, 2$ and $\beta_i = s_i g_i$.



$Q_{i,j} = O$ if $0 \le i \le g_1 - 1$ or $0 \le j \le g_2 - 1$

$Q_{i,j} = E_{u_i,w_j} - E_{i,w_j} - E_{u_i,j} + E_{i,j}$

$$
\begin{aligned}
Q_{i,j} &= \sum_{m=0}^{\zeta_i-1} \left(C_{u_i+m\beta_1,j} - C_{u_i+m\beta_1+1,j} \right) \\
&= \sum_{m=0}^{\zeta_i-1} \Bigg[\sum_{a=0}^{\beta_1-1} (E_{u_i+m\beta_1+a,w_j} - E_{u_i+m\beta_1+a,j}) \\
&\qquad\qquad - \sum_{a=0}^{\beta_1-1} (E_{u_i+m\beta_1+a+1,w_j} - E_{u_i+m\beta_1+a+1,j}) \Bigg] \\
&= \sum_{m=0}^{\zeta_i-1} \Bigg[\sum_{a=0}^{\beta_1-1} (E_{u_i+m\beta_1+a,w_j} - E_{u_i+m\beta_1+a+1,w_j}) \\
&\qquad\qquad - \sum_{a=0}^{\beta_1-1} (E_{u_i+m\beta_1+a,j} - E_{u_i+m\beta_1+a+1,j}) \Bigg] \\
&= \sum_{m=0}^{\zeta_i-1} (E_{u_i+m\beta_1,w_j} - E_{u_i+(m+1)\beta_1,w_j}) - \sum_{m=0}^{\zeta_i-1} (E_{u_i+m\beta_1,j} - E_{u_i+(m+1)\beta_1,j}) \\
&= E_{u_i,w_j} - E_{u_i+\zeta_i\beta_i,w_j} - E_{u_i,j} + E_{u_i+\zeta_i\beta_i,j} \\
&= E_{u_i,w_j} - E_{i,w_j} - E_{u_i,j} + E_{i,j}.
\end{aligned}
$$

Functions $\mathcal{T}$ and $\mathcal{S}$

$Q_{i,j} = O$ if $0 \leq i \leq g_1 - 1$ or $0 \leq j \leq g_2 - 1$

Definition of the function $\mathcal{T}$

$$\mathcal{T}(A) := A - \sum_{j=0}^{\alpha_2 - 1} \sum_{i=0}^{\alpha_1 - 1} [A]_{i,j} Q_{i,j}$$

$\mathcal{T}(A) := A - \sum_{j=0}^{\alpha_2 - 1} \sum_{i=0}^{\alpha_1 - 1} [A]_{i,j} Q_{i,j}$

$[T(A)]_{i,j} = 0$ if $g_1 \leq i$ and $g_2 \leq j$.

For example, in the $((6,8); (2,2); 5)$-DR game, $g_1 = \gcd(6,2) = 2$, $g_2 = \gcd(8,2) = 2$, $s_1 = s_2 = 1$. Let

$A =$

1	1	0	4	4	1	1	0
2	2	1	0	1	2	1	1
1	1	1	0	1	1	3	4
0	2	2	0	0	1	4	3
2	2	3	1	0	1	1	2
2	0	1	0	4	0	0	2

.

$A - [A]_{2,6} Q_{2,6} =$

1	1	0	4	4	1	1	0
2	2	1	0	1	2	1	1
1	1	1	0	1	1	3	4
0	2	2	0	0	1	4	3
2	2	3	1	0	1	1	2
2	0	1	0	4	0	0	2

$- 3 \times$

1	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0
-1	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

.

$\mathcal{T}(A) := A - \sum_{j=0}^{\alpha_2-1} \sum_{i=0}^{\alpha_1-1} [A]_{i,j} Q_{i,j}$

$[T(A)]_{i,j} = 0$ if $g_1 \leq i$ and $g_2 \leq j$.

For example, in the $((6,8);(2,2);5)$-DR game, $g_1 = \gcd(6,2) = 2$, $g_2 = \gcd(8,2) = 2$, $s_1 = s_2 = 1$. Let

$A =$

1	1	0	4	4	1	1	0
2	2	1	0	1	2	1	1
1	1	1	0	1	1	3	4
0	2	2	0	0	1	4	3
2	2	3	1	0	1	1	2
2	0	1	0	4	0	0	2

.

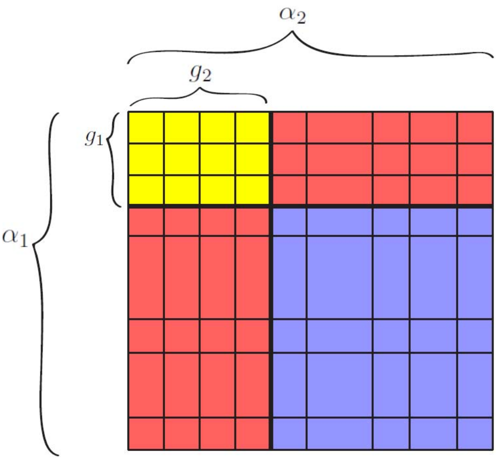
$\mathcal{T}(A) = A - \sum_{j=2}^{7} \sum_{i=2}^{5} [A]_{i,j} Q_{i,j} =$

2	2	4	0	0	3	0	1
1	1	4	0	0	3	0	1
1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	0	0	0	0	0	0

.

Concluding remarks

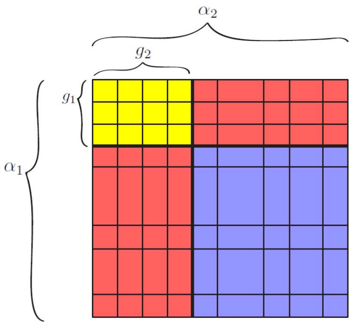
$\mathcal{T}(A) := A - \sum_{j=0}^{\alpha_2-1} \sum_{i=0}^{\alpha_1-1} [A]_{i,j} Q_{i,j}$



Functions $\mathcal{T}$ and $\mathcal{S}$

Definition of the function $\mathcal{S}$

$$\mathcal{S}(A) := \mathcal{T}(A) + \sum_{i=0}^{\alpha_1-1} [\mathcal{T}(A)]_{i,0} \frac{1}{s_2} \mathcal{T}(R_{i,0}) + \sum_{j=0}^{\alpha_2-1} [\mathcal{T}(A)]_{0,j} \frac{1}{s_1} \mathcal{T}(C_{0,j}).$$



$$\mathcal{S}(A) := \mathcal{T}(A) + \sum_{i=0}^{\alpha_1-1} [\mathcal{T}(A)]_{i,0} \frac{1}{s_2} \mathcal{T}(R_{i,0}) + \sum_{j=0}^{\alpha_2-1} [\mathcal{T}(A)]_{0,j} \frac{1}{s_1} \mathcal{T}(C_{0,j}).$$

$\mathcal{T}(A) =$

2	2	4	0	0	3	0	1
1	1	4	0	0	3	0	1
1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	0	0	0	0	0	0

.

$$\mathcal{S}(A) = A - \sum_{j=2}^{7} \sum_{i=2}^{5} [A]_{i,j} Q_{i,j} + 4\mathcal{T}(C_{0,2}) + 3\mathcal{T}(C_{0,5}) + \mathcal{T}(C_{0,7}) + \mathcal{T}(R_{2,0}) + \mathcal{T}(R_{3,0}) + \mathcal{T}(R_{4,0}) + 2\mathcal{T}(R_{5,0})$$

$=$

3	3	0	0	0	0	0	0
3	3	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$= 3J^*_{0,0}.$

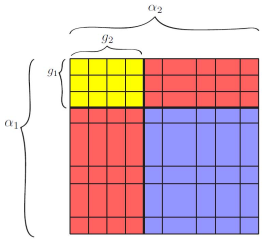
Characterizing the solutions

Theorem (Main)

A matrix A in $\mathcal{M}((\alpha_1,\alpha_2),n)$ *is a solution matrix of the* $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$*-DR game if and only if for some* d_i, e_j, $t \in \mathbb{Z}_n$,

$$\mathcal{T}(A) = s_2 \sum_{i=0}^{\alpha_1-1} d_i J_{i,0}^{(1,g_2)} + s_1 \sum_{j=0}^{\alpha_2-1} e_j J_{0,j}^{(g_1,1)} - s_1 s_2 t J_{0,0}^{(g_1,g_2)}$$

$$\mathcal{S}(A) = t s_1 s_2 J_{0,0}^{(g_1,g_2)}$$



Example

For example, in the $((6,8);(2,2);5)$-DR game, $g_1 = \gcd(6,2) = 2$, $g_2 = \gcd(8,2) = 2$, $s_1 = s_2 = 1$. Let

$A =$

1	1	0	4	4	1	1	0
2	2	1	0	1	2	1	1
1	1	1	0	1	1	3	4
0	2	2	0	0	1	4	3
2	2	3	1	0	1	1	2
2	0	1	0	4	0	0	2

.

$\mathcal{T}(A) = A - \sum_{j=2}^{7} \sum_{i=2}^{5} [A]_{i,j} Q_{i,j} =$

2	2	4	0	0	3	0	1
1	1	4	0	0	3	0	1
1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	0	0	0	0	0	0

.

Example

$$\mathcal{S}(A) = A - \sum_{j=2}^{7} \sum_{i=2}^{5} [A]_{i,j} Q_{i,j} + 4\mathcal{T}(C_{0,2}) + 3\mathcal{T}(C_{0,5}) + \mathcal{T}(C_{0,7}) + \mathcal{T}(R_{2,0}) + \mathcal{T}(R_{3,0}) + \mathcal{T}(R_{4,0}) + 2\mathcal{T}(R_{5,0})$$

=

3	3	0	0	0	0	0	0
3	3	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$= 3J^{*}_{0,0}.$

Therefore, A is a solution matrix by Main Theorem.

Example

$$\mathcal{T}(A) := A - \sum_{j=0}^{\alpha_2-1} \sum_{i=0}^{\alpha_1-1} [A]_{i,j} Q_{i,j}$$

$$\mathcal{S}(A) := \mathcal{T}(A) + \sum_{i=0}^{\alpha_1-1} [\mathcal{T}(A)]_{i,0} \frac{1}{s_2} \mathcal{T}(R_{i,0}) + \sum_{j=0}^{\alpha_2-1} [\mathcal{T}(A)]_{0,j} \frac{1}{s_1} \mathcal{T}(C_{0,j}).$$

$$A - \sum_{j=0}^{\alpha_2-1} \sum_{i=0}^{\alpha_1-1} [A]_{i,j} Q_{i,j} + \sum_{i=0}^{\alpha_1-1} [\mathcal{T}(A)]_{i,0} \frac{1}{s_2} \mathcal{T}(R_{i,0}) + \sum_{j=0}^{\alpha_2-1} [\mathcal{T}(A)]_{0,j} \frac{1}{s_1} \mathcal{T}(C_{0,j}) - \mathcal{S}(A) = O$$

Computing a solving coefficient matrix

$$A - \sum_{j=2}^{7} \sum_{i=2}^{5} [A]_{i,j} Q_{i,j} + 4\mathcal{T}(C_{0,2}) + 3\mathcal{T}(C_{0,5}) + \mathcal{T}(C_{0,7}) + \mathcal{T}(R_{2,0}) + \mathcal{T}(R_{3,0}) + \mathcal{T}(R_{4,0}) + 2\mathcal{T}(R_{5,0}) - 3J^*_{0,0} = O.$$

Example

Then a solving coefficient matrix of $((6,8);(2,2);5)$-DR game corresponding to A is

$$\begin{pmatrix} 4 & 0 & 2 & 1 & 0 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 & 3 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 & 0 & 4 & 0 & 0 \\ 3 & 2 & 2 & 3 & 3 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1-dimensional case

Circles which are solution of dice rolling game

When $\alpha_1 = \beta_1 = 1$, a matrix in $\mathcal{M}((\alpha_1, \alpha_2), n)$ becomes an α_2-tuple, which are circles on each of which α_2 n-dice are located.

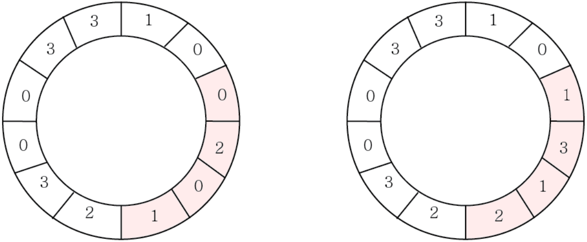


Figure: Circles in $((1,12);(1,4);5)$-DR game

1-dimensional case

Case that $\alpha_1 = \beta_1$

$R_{i,j} = O$ if $0 \le i \le g_1 - 1$
$C_{i,j} = O$ if $0 \le j \le g_2 - 1$
$Q_{i,j} = O$ if $0 \le i \le g_1 - 1$ or $0 \le j \le g_2 - 1$

$$\text{If } \alpha_1 = \beta_1, \quad \mathcal{T}(A) = A, \quad \mathcal{S}(A) = A + \sum_{j=0}^{\alpha_2 - 1} [A]_{0,j} C_{0,j}.$$

Suppose that $\alpha_1 = \beta_1$. Then $g_1 = \gcd(\alpha_1, \beta_1) = \alpha_1$ and so $s_1 = 1$.

Case that $\alpha_1 = \beta_1$

A matrix A is a solution matrix of the $((\alpha_1, \alpha_2); (\alpha_1, \beta_2); n)$-DR game if and only if

$$A = s_2 \sum_{i=0}^{\alpha_1 - 1} d_i J_{(i,0)}^{(1,g_2)} + \sum_{j=0}^{\alpha_2 - 1} e_j J_{0,j}^{(g_1,1)} - s_2 t J_{0,0}^{(\alpha_1, g_2)}$$

for some d_i, e_j, $t \in \mathbb{Z}_n$ and $\mathcal{S}(A) = u s_2 J_{(0,0)}^{(\alpha_1, g_2)}$ for some $u \in \mathbb{Z}_n$.

1-dimensional case

Case that $\alpha_1 = \beta_1$

If $\mathcal{S}(A) = us_2 J_{(0,0)}^{(\alpha_1, g_2)}$ for some $u \in \mathbb{Z}_n$, then it holds that

$$A = -\mathcal{S}(A) + \sum_{j=0}^{\alpha_2 - 1} [A]_{0,j} C_{0,j} = -us_2 J_{(0,0)}^{(\alpha_1, g_2)} + \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} [A]_{0,j} (E_{i,w_j} - E_{i,j})$$

$$= \sum_{j=0}^{g_2 - 1} (-us_2) \sum_{i=0}^{\alpha_1 - 1} E_{i,j} + \sum_{j=0}^{\alpha_2 - 1} [A]_{0,j} \sum_{i=0}^{\alpha_1 - 1} (E_{i,w_j} - E_{i,j})$$

$$= \sum_{j=0}^{g_2 - 1} (-us_2) \sum_{i=0}^{\alpha_1 - 1} E_{i,j} + \sum_{j=0}^{\alpha_2 - 1} [A]_{0,j} \sum_{i=0}^{\alpha_1 - 1} (E_{i,w_j} - E_{i,j}) = \sum_{j=0}^{\alpha_2 - 1} f_j \sum_{i=0}^{\alpha_1 - 1} E_{i,j}$$

for some $f_j \in \mathbb{Z}_n$. Therefore for some $f_j \in \mathbb{Z}_n$,

$$A = s_2 \sum_{i=0}^{\alpha_1 - 1} \left(0 \cdot \sum_{j=0}^{g_2 - 1} E_{i,j} \right) + \sum_{j=0}^{\alpha_2 - 1} \left(f_j \sum_{i=0}^{\alpha_1 - 1} E_{i,j} \right) - s_2 \cdot 0 \cdot J_{0,0}^{(\alpha_1, g_2)}$$

1-dimensional case

Case that $\alpha_1 = \beta_1$

A matrix A is a solution matrix of the $((\alpha_1, \alpha_2); (\alpha_1, \beta_2); n)$-DR game if and only if

$$\mathcal{S}(A) = us_2 J_{(0,0)}^{(\alpha_1, g_2)}$$

for some $u \in \mathbb{Z}_n$.

Circles which are solution of dice rolling game

An ordered α_2-tuple $\mathbf{v}$ is a solution matrix of the $((1, \alpha_2); (1, \beta_2); n)$-DR game if and only if $\mathcal{S}(\mathbf{v}) = (\underbrace{us_2, \ldots, us_2}_{g_2}, 0, \ldots, 0)$ for some $u \in \mathbb{Z}_n$.

The 2-dimensional case can be generalized to the t-dimensional case for $t \geq 1$ if we can find a way to use notations more efficiently.

1-dimensional case

Case that $\alpha_1 = \beta_1$

A matrix A is a solution matrix of the $((\alpha_1, \alpha_2); (\alpha_1, \beta_2); n)$-DR game if and only if

$$\mathcal{S}(A) = us_2 J_{(0,0)}^{(\alpha_1, g_2)}$$

for some $u \in \mathbb{Z}_n$.

Circles which are solution of dice rolling game

An ordered α_2-tuple $\mathbf{v}$ is a solution matrix of the $((1, \alpha_2); (1, \beta_2); n)$-DR game if and only if $\mathcal{S}(\mathbf{v}) = (\underbrace{us_2, \ldots, us_2}_{g_2}, 0, \ldots, 0)$ for some $u \in \mathbb{Z}_n$.

The 2-dimensional case can be generalized to the t-dimensional case for $t \geq 1$ if we can find a way to use notations more efficiently.

Thank you very much.