Semilattice Polymorphisms on Reflexive Graphs

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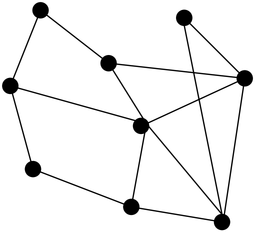
Kyungpook National University

August 21, 2009

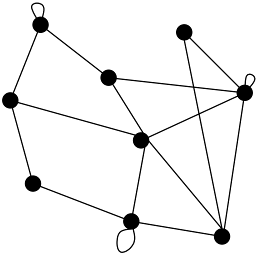
Outline

- Polymorphisms
  - why we care about them
  - what are they
- Reflexive Graphs
- Semilattice Polymorphisms
- Semilattice Polymorphisms on Reflexive Graphs
- Chordal Reducible Graphs

Polymorphisms (why we care about them)



For every relational structure H



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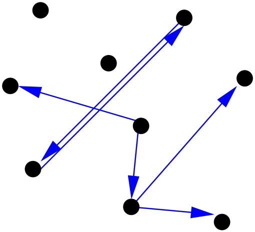

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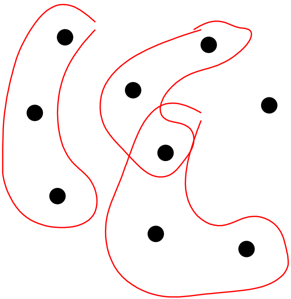

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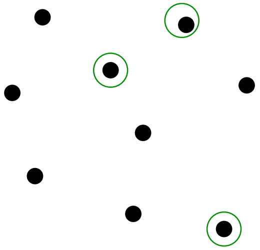

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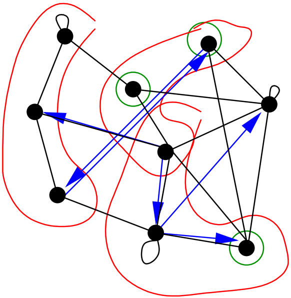

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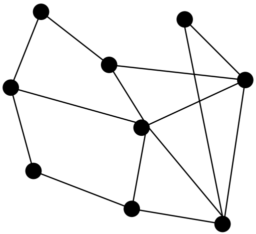

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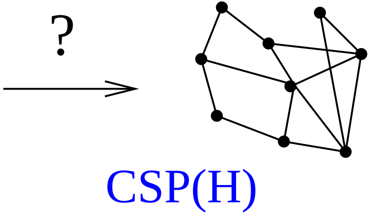
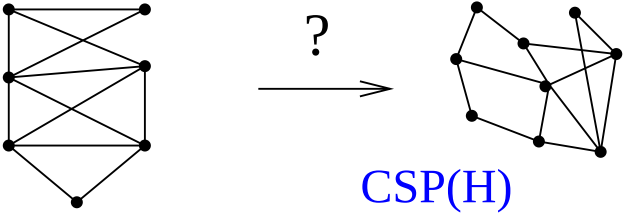


For every relational structure H there is a computational problem $\mathrm{CSP}(H)$.



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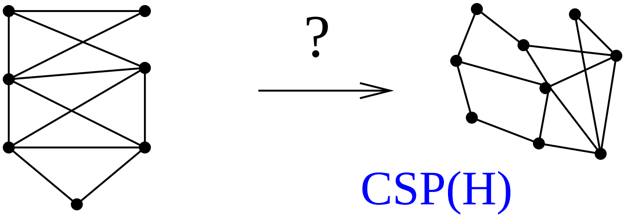
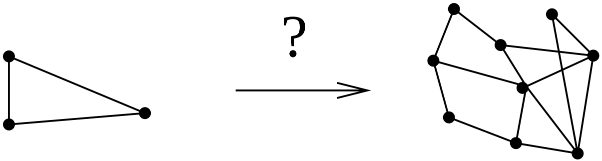


A homomorphism $\phi : G \to H$ is a vertex map that preserves relations.



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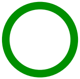

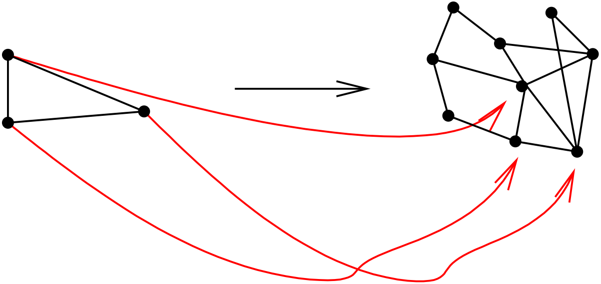

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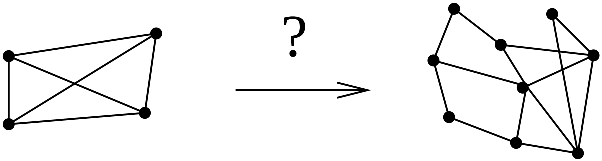

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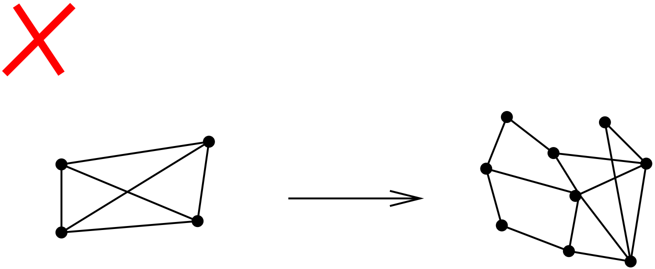

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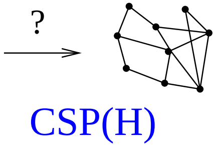
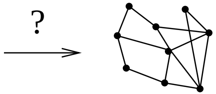
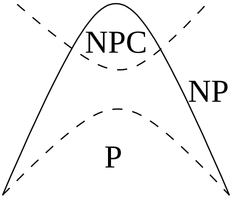


We are interested in the computational complexity of $\mathrm{CSP}(H)$.



CSP(H)

Complexity



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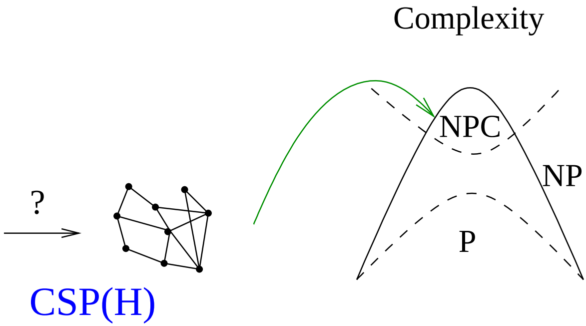

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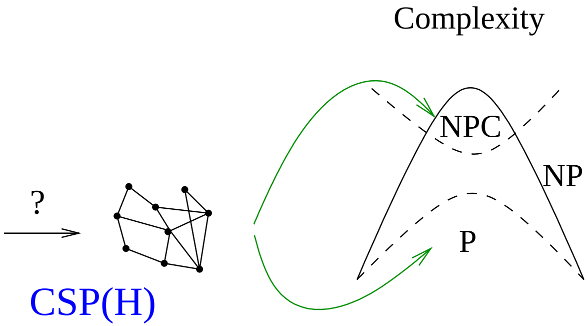

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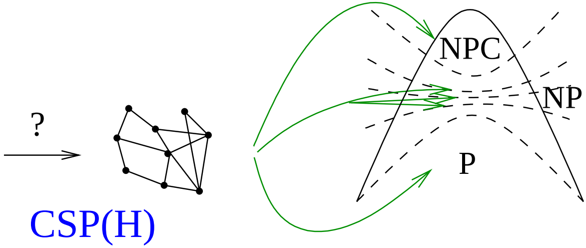

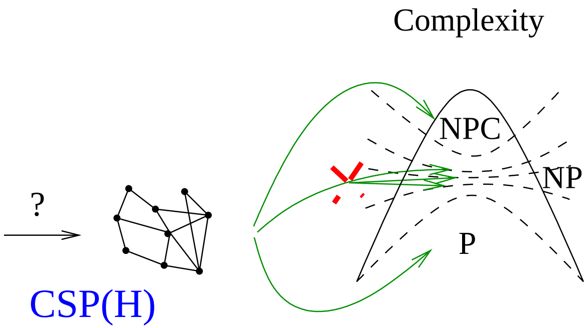

CSP Dichotomy Conjecture [Feder, Vardi '99]

For any H, $\mathrm{CSP}(H)$ is in either P or NPC.

The CSP Dichotomy Conjecture is true ...

- for structures on two vertices. Schaefer '78
- for graphs. Hell, Nešetřil '92
- for structures on three vertices. Bulatov '02
- for conservative structures (list-colouring). Bulatov '06
- for digraphs without sources or sinks. Barto, Kozik, Niven '09

Theorem Jeavons '00

The complexity of $\mathrm{CSP}(H)$ is determined by the polymorphisms of H.

Polymorphisms (what are they)

Definition: Polymorphism

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$\phi : V(H) \times \cdots \times V(H) \to V(H)$

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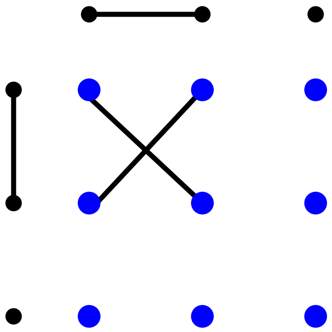

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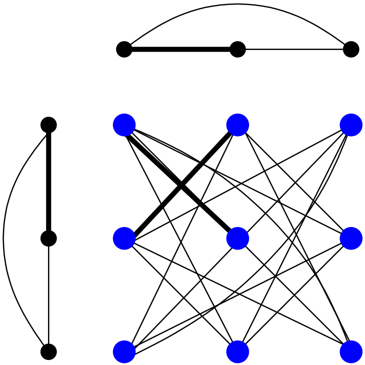
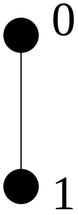


Example: The 3-ary polymorphisms of K_2

$\mathrm{Pol}(K_2)$



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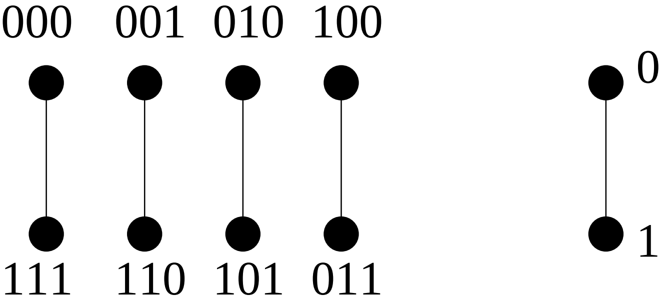

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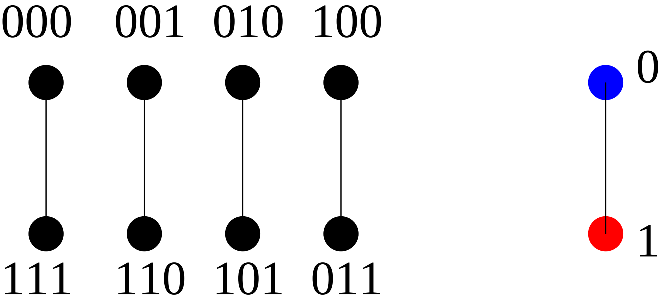

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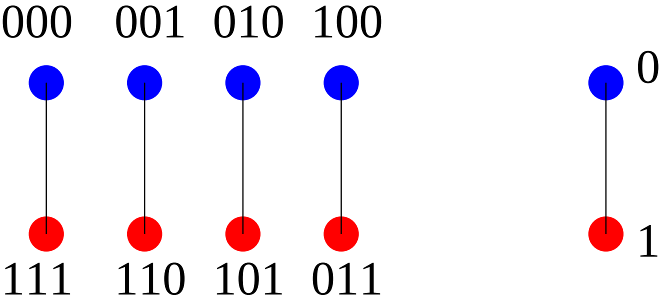

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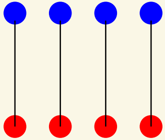

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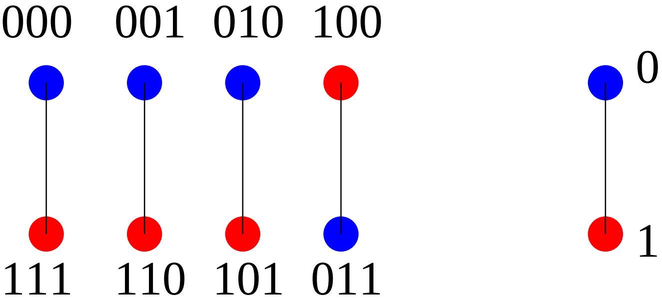

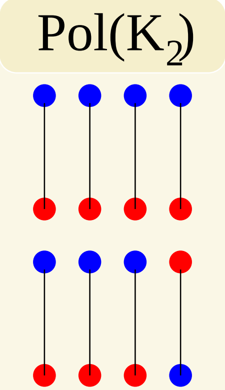

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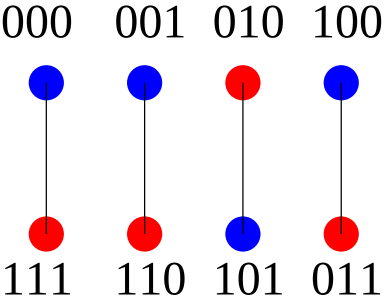

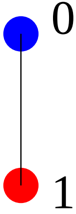

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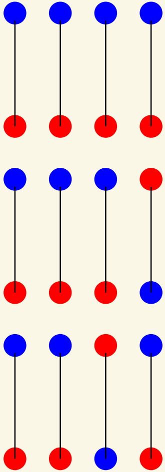

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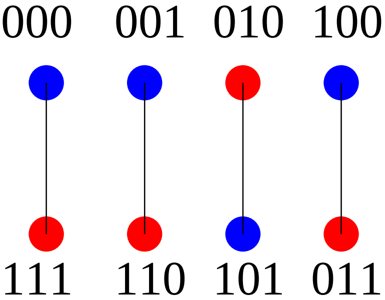
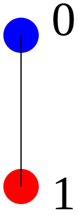
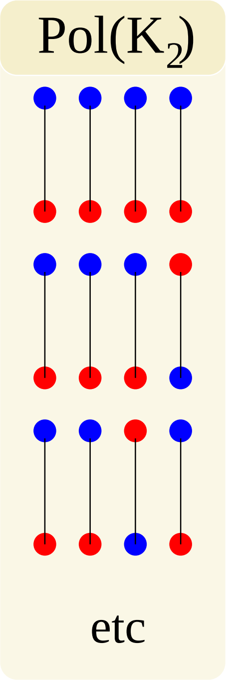
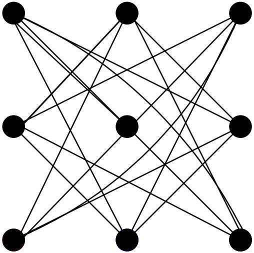


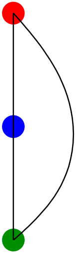


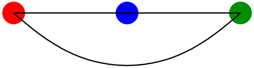


Example: The 2-ary polymorphisms of K_3

Pol(K_3)







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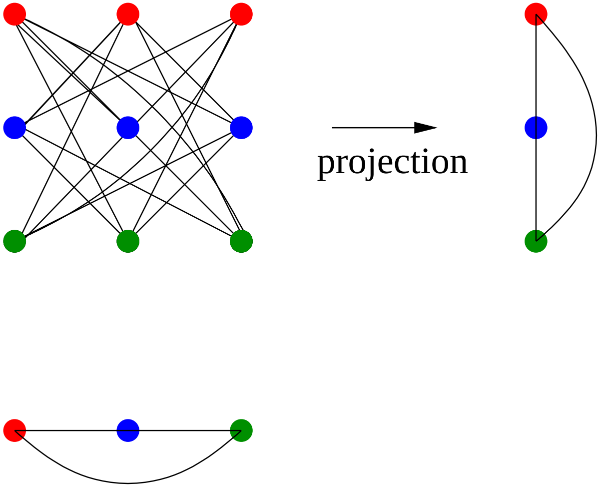

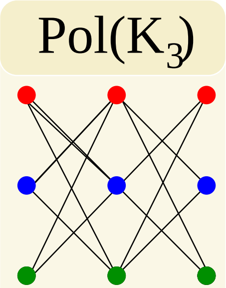

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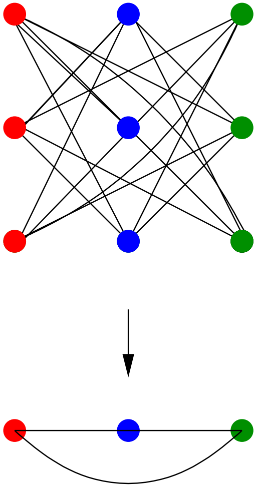

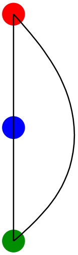

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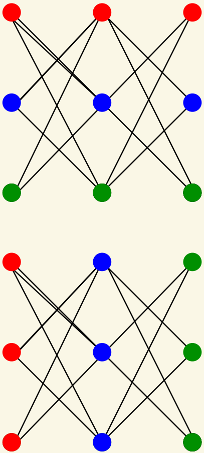

Theorem Jeavons '00

If $\mathrm{Pol}(H)$ contains only projections, then $\mathrm{CSP}(H)$ is in NPC.

A polymorphism $\phi : H^d \to H$ is

WNU (weak near-unanimity)

if

$$\begin{aligned} \phi(x, x, \ldots, x, y) &= \phi(x, x, \ldots, y, x) = \ldots \\ &= \phi(y, x, \ldots, x, x) \end{aligned}$$

for all $x, y \in V(H)$.

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Conjecture: [BJK'02; MM'08]

CSP(H) is in NPC if H admits no WNU polymorphisms, and is otherwise polynomial time solvable.

BJK: Bulatov, Jeavons, Krokhin
MM: Maroti, McKenzie

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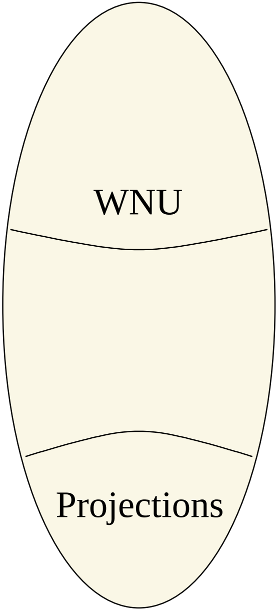

A polymorphism $\phi : H^d \to H$ is

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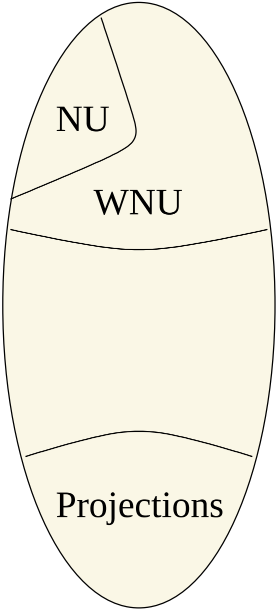

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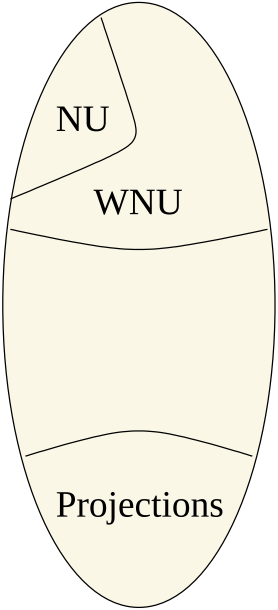
If H admits an NU polymorphism, then CSP(H) is polynomial time solvable.



A polymorphism $\phi : H^d \to H$ is

TSI (totally symmetric idempotent)

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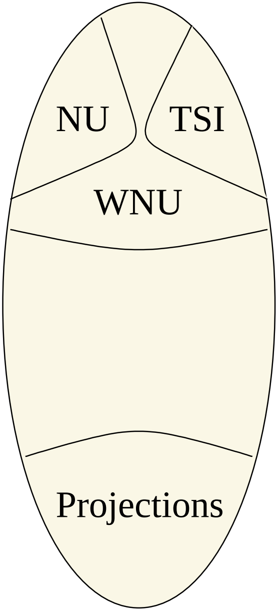
whenever $\{u_1, \ldots, u_d\} = \{v_1, \ldots, v_d\}$ as sets.



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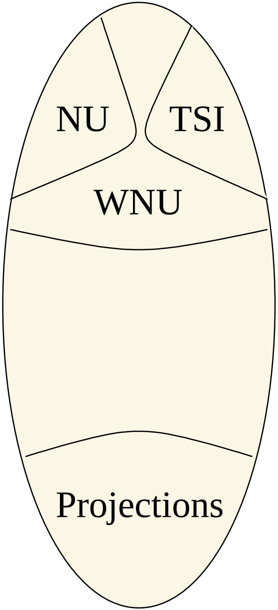
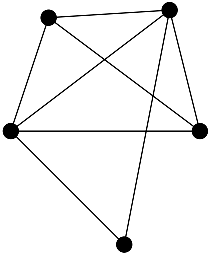
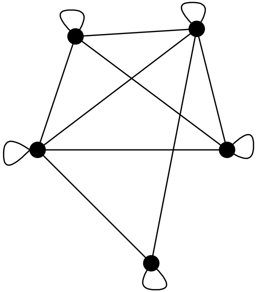
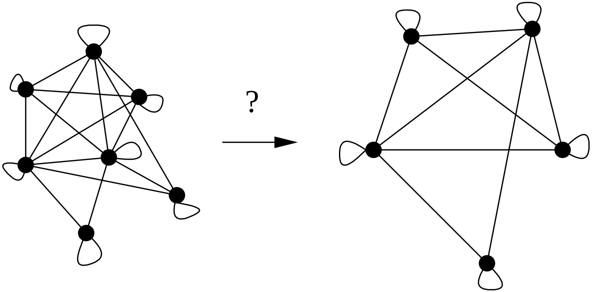
If H admits a TSI polymorphism, then CSP(H) is polynomial time solvable.

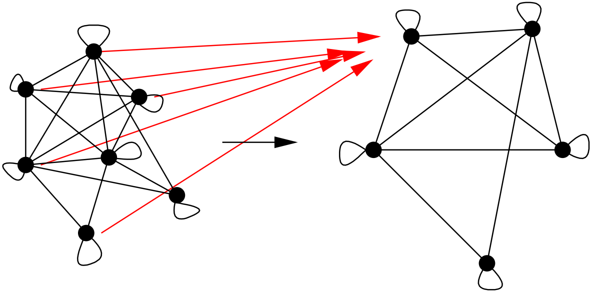


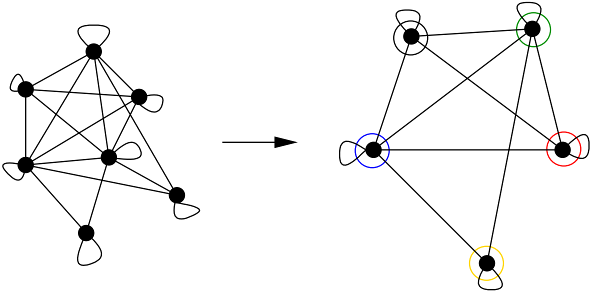
Reflexive Graphs

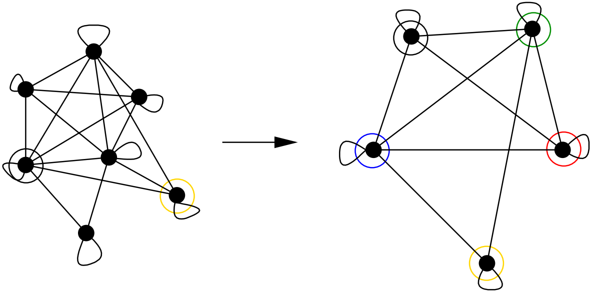


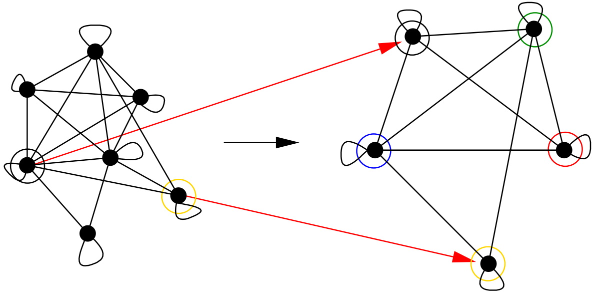






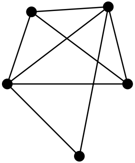




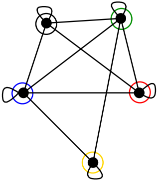


Assume all graphs are connected, reflexive and have all singleton unary relations.

We draw



to mean



Why Reflexive Graphs?

- Dichotomy is done for irreflexive graphs, and hard for digraphs. Reflexive graphs are somewhere in between.
- Dichotomy is done for MinHOM of reflexive graphs. [GHRY '07]. (Infact for digraphs with possible loops.)
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GHRY: Gutin Hell Rafiey Yeo

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BFHHM: Brewster Feder Hell Huang MacGillivray;
LLT: Larose Loten Tardif

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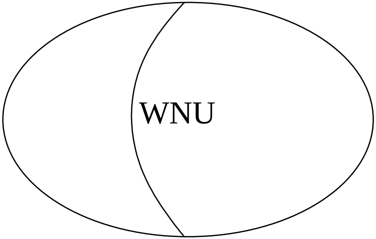
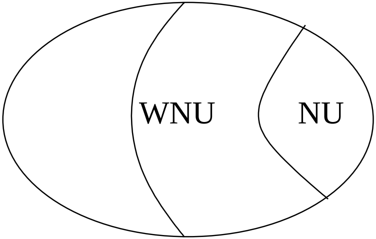


Towards dichotomy on reflexive graphs, we want to know what graphs admit WNU.

Reflexive Graphs



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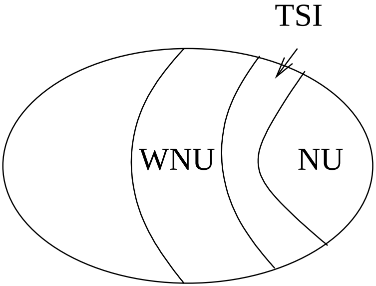
Reflexive Graphs



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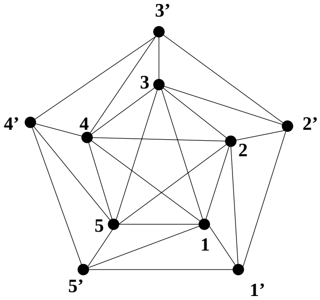
Goals

- Characterise reflexive graphs admitting TSI of all arities.
- Characterise reflexive graphs admitting TSI.
- Characterise reflexive graphs admitting WNU.
- Prove Dichotomy for reflexive graphs.

Semilattice Polymorphisms



Let ϕ be defined by

- idempotence.
- maximality on non-primed vertices (ties t
  min label)
- for mix of primed and non-primed entries
  ignore the primed entries and do as in th
  previous step.
- If all entries are primed then
  - if they are i' and $(i+1)'$, go to $i +$
  - if they are $(i-1)'$ and $(i+1)'$, go t
    i
  - if they are $(i-1)'$, $(i)'$ and $(i+1)'$
    go to i
  - otherwise, remove their primes (ie,
    read i' as i) and go to the min entr

Definition

A 2-ary polymorphism $\phi : H^2 \to H$ is SL (semilattice) if it is idempotent, associative and commutative.

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Such an operation is called semilattice because the partial ordering

$$u < v \text{ if } \phi(u, v) = u$$

of $V(H)$ is a meet semilattice .

Definition

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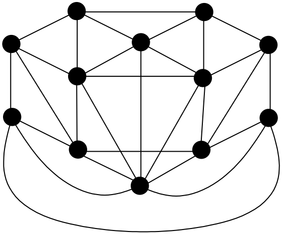
$$u < v \text{ if } \phi(u, v) = u$$

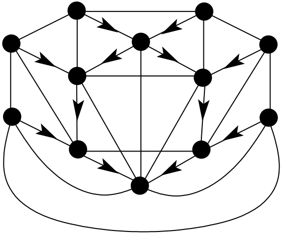
Where $\wedge$ is the associated meet, we have

$$\phi(u, v) = u \wedge v,$$

so we will denote SL polymorphisms by $\wedge$.



Semilattice Polymorphisms are easy to represent.



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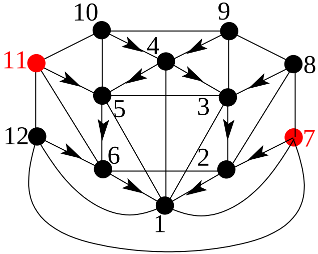

$11 \wedge 7$

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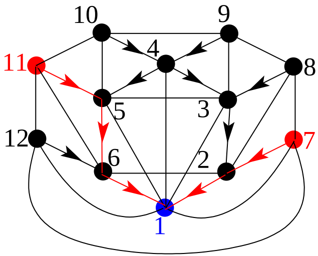
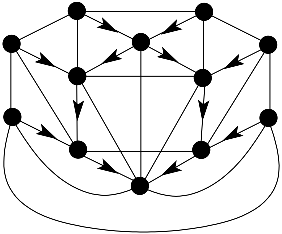


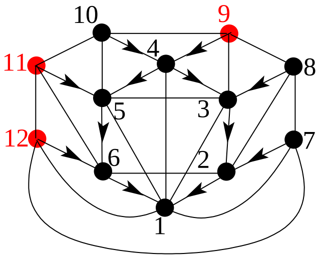
$11 \wedge 7 = 1$

Semilattice Polymorphisms are easy to represent.



$$\phi : (v_1, \ldots, v_d) \mapsto v_1 \wedge \cdots \wedge v_d$$

And define TSI of every arity.



$\phi(9, 11, 12)$

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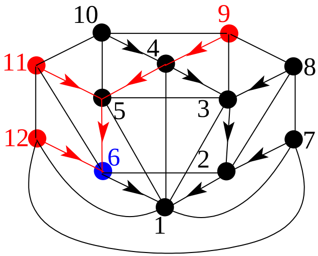
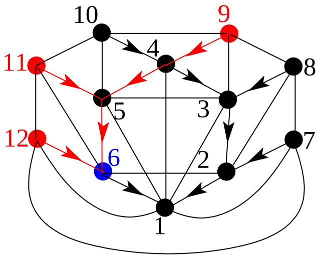
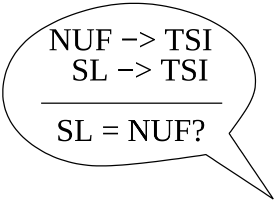


$\phi(9, 11, 12) = 9 \wedge 11 \wedge 12 = 6$

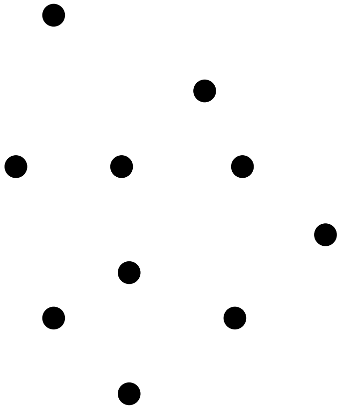
And define TSI of every arity.



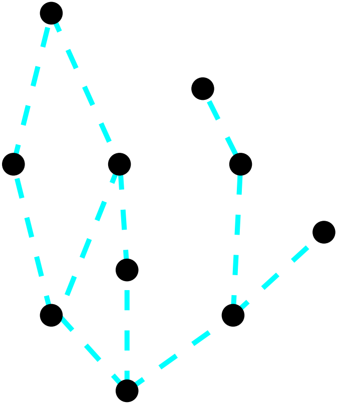


And define TSI of every arity.

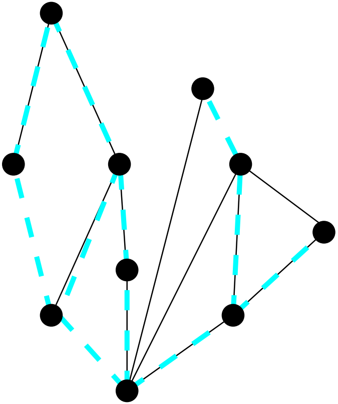
Semilattice Polymorphisms on Reflexive Graphs



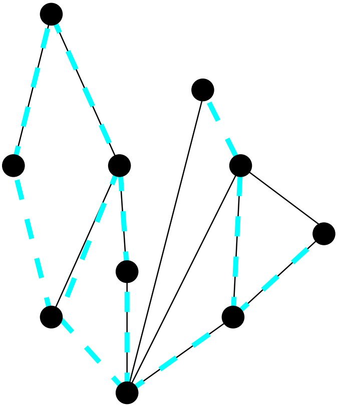
Given some vertices,



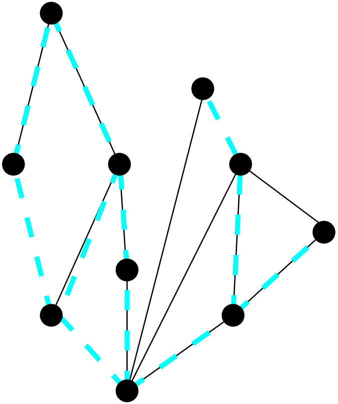
Given some vertices, a semilattice ordering,



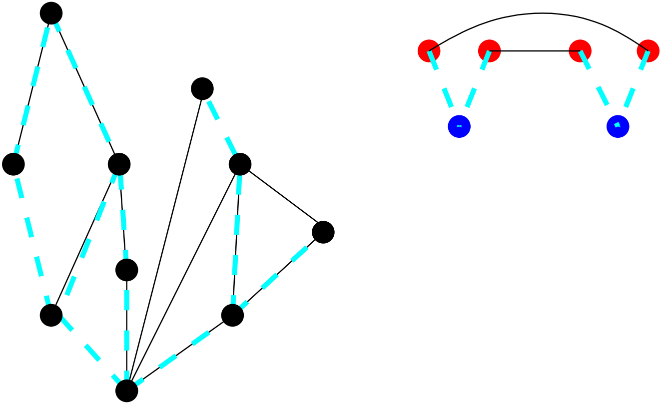
Given some vertices, a semilattice ordering, and a reflexive graph on the vertices,



Given some vertices, a semilattice ordering, and a reflexive graph on the vertices, Is the semilattice *polymorphic?*



Polymorphism: $u \sim u', v \sim v' \Rightarrow u \wedge v \sim u' \wedge v'$



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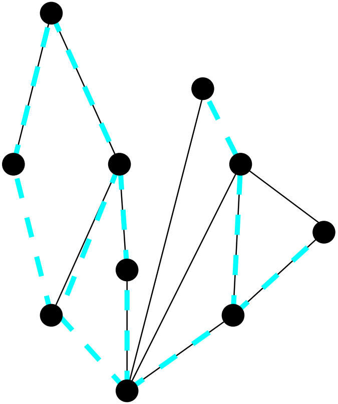
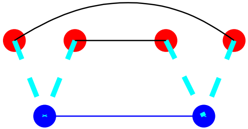
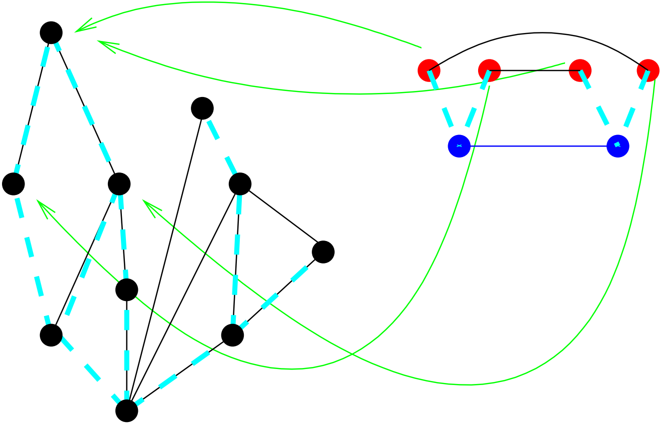
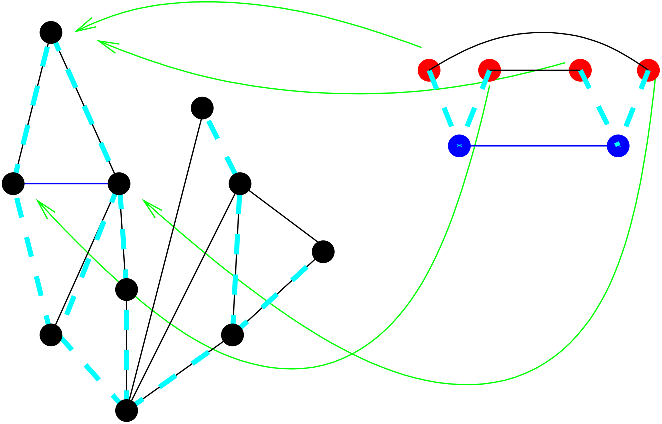
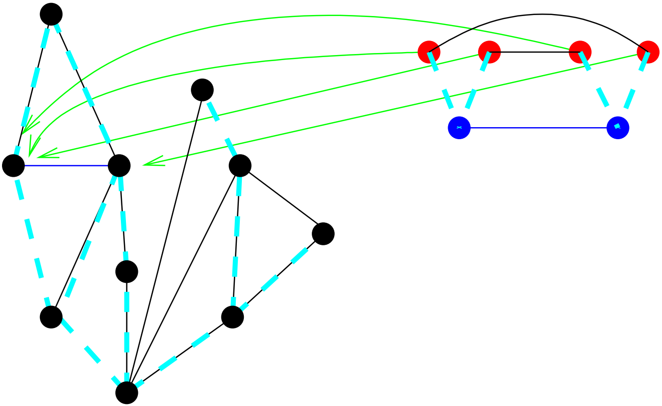
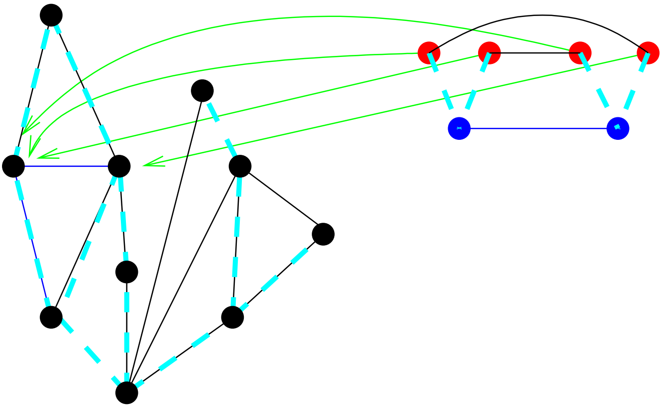


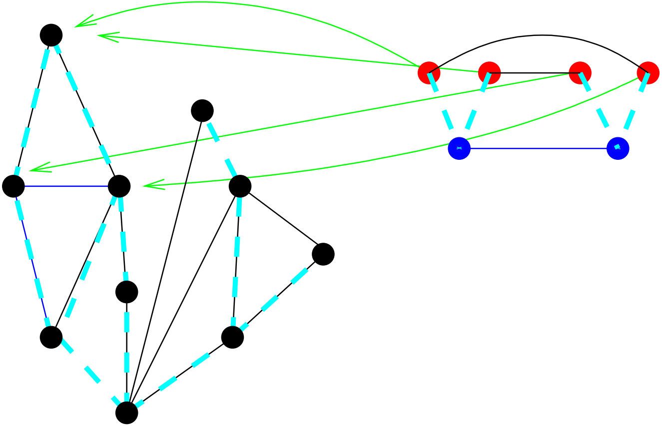


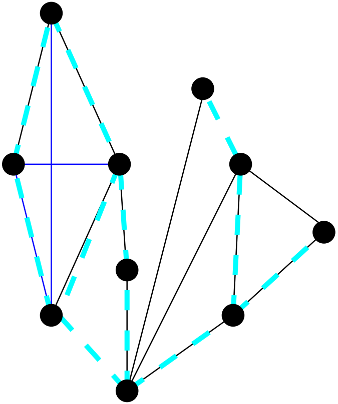


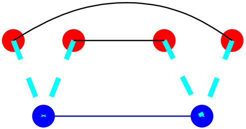


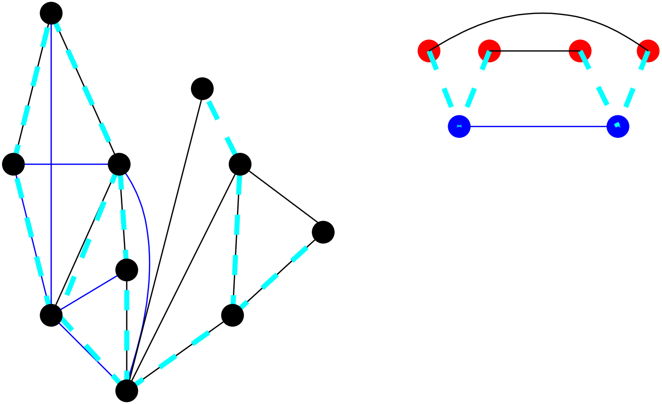


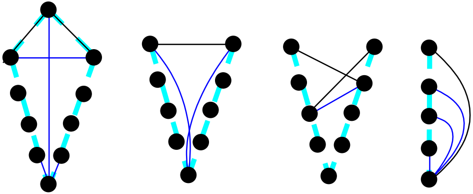




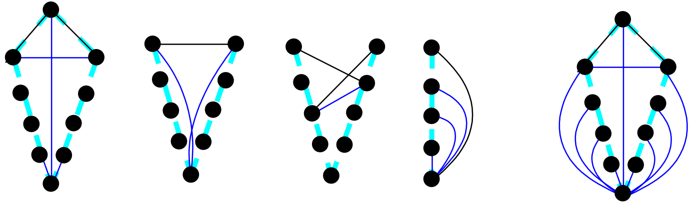








Consequential identities.



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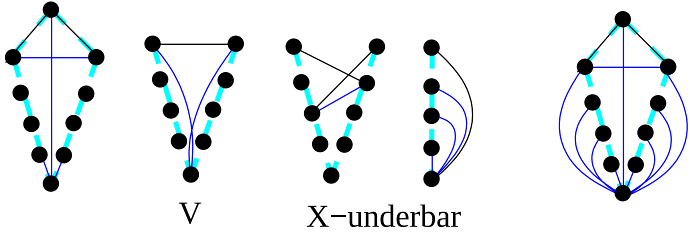
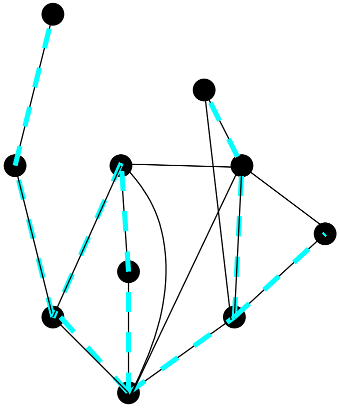


Consequential identities.

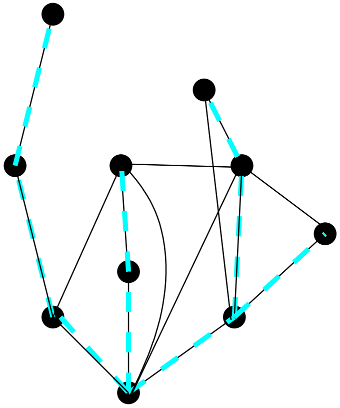
Types of Semilattice Polymorphisms



A semilattice polymorphism is ...

- embedded if every *Hasse* edge (blue edge) is a graph edge.

Types of Semilattice Polymorphisms



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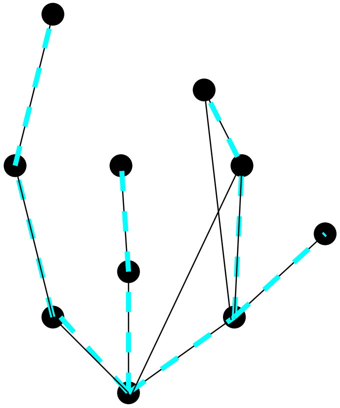
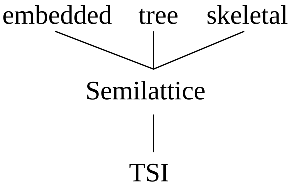


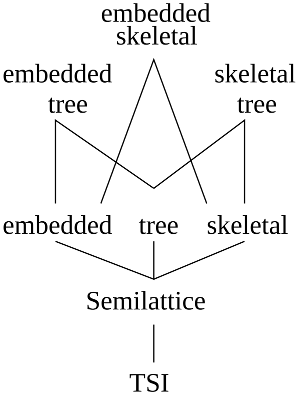
A semilattice polymorphism is ...

- embedded if every *Hasse* edge (blue edge) is a graph edge.
- tree if the Hasse edges induce a tree.
- skeletal if all graph edges are between comparible vertices.

Semilattice

TSI


embedded
tree
skeletal
Semilattice
TSI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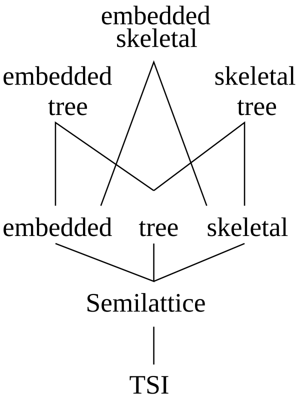
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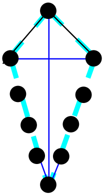

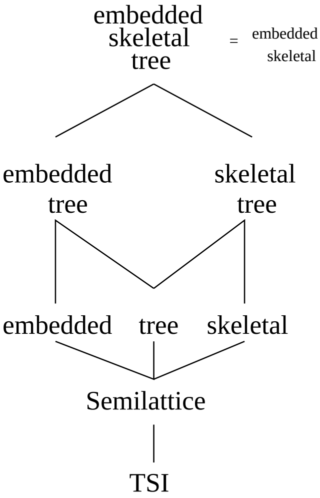
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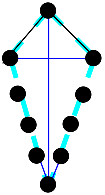
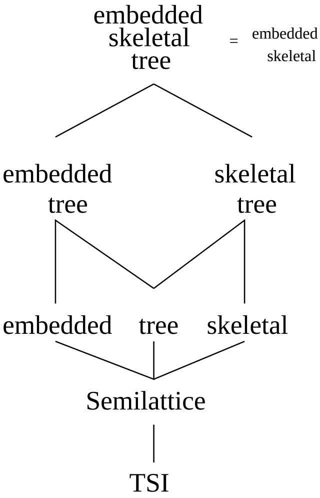
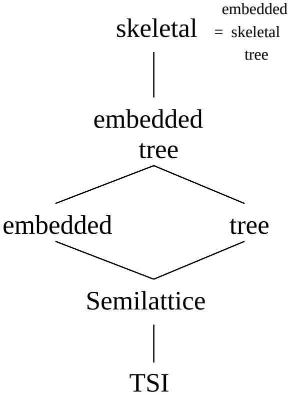




H admits a skeletal SL

$\Rightarrow$

H admits an embedded skeletal tree SL



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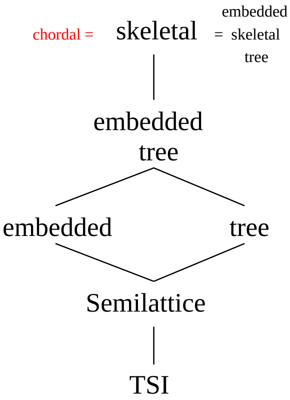


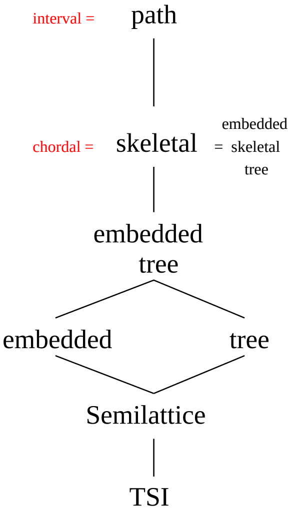
H admits a skeletal SL

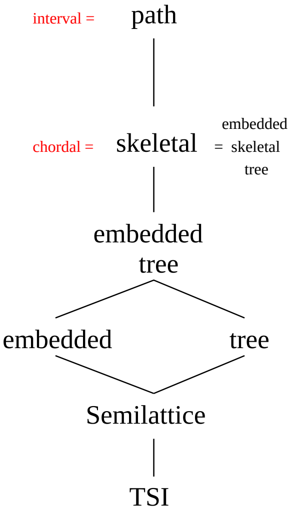
$\Rightarrow$

H is chordal

$\Rightarrow$

H admits an embedded skeletal tree SL


interval =
path
chordal =
skeletal
embedded
= skeletal
tree
embedded
tree
embedded
tree
Semilattice
TSI

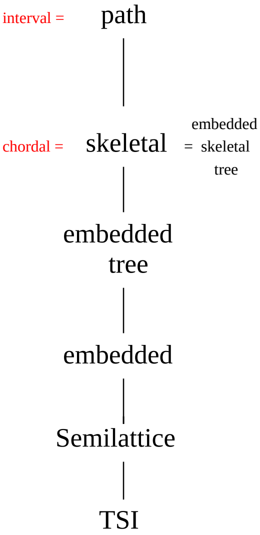


Proposition

H admits a tree SL,

$$\Rightarrow$$

H admits an embedded tree SL.



Proposition

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$$\Rightarrow$$

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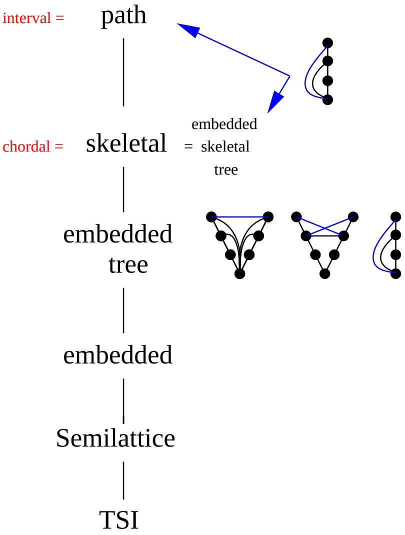
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chordal =
skeletal
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= skeletal
tree
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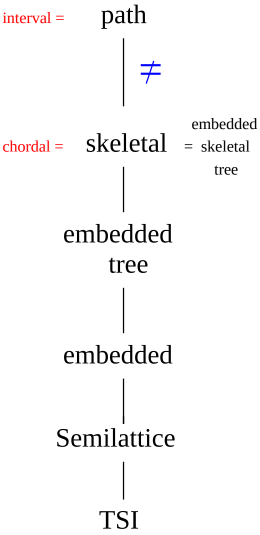
tree
embedded
Semilattice
TSI


interval =
path
≠
chordal =
skeletal
= embedded skeletal tree
embedded tree
embedded
Semilattice
TSI

interval = path

$\neq$

chordal = skeletal = embedded skeletal tree

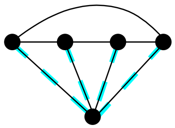
$\neq$

embedded tree

embedded

Semilattice

TSI



interval = path

≠

chordal = skeletal = embedded skeletal tree

≠

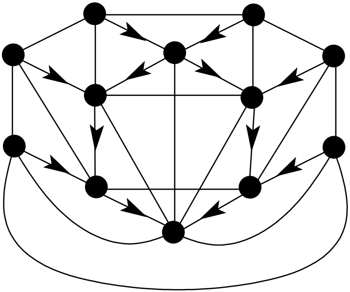
embedded tree

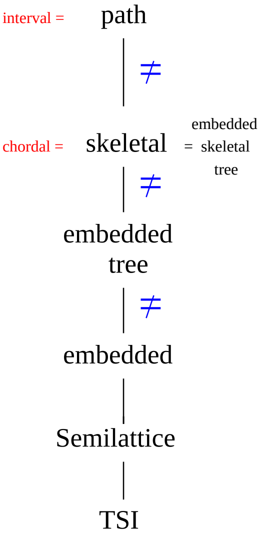
≠

embedded

Semilattice

TSI




interval =
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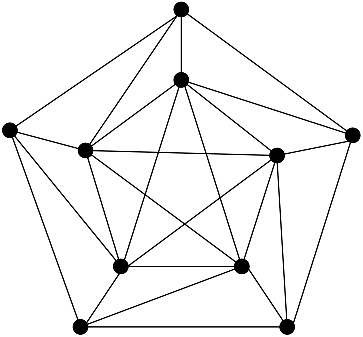


interval = path

≠

chordal = skeletal = embedded skeletal tree

≠

embedded tree

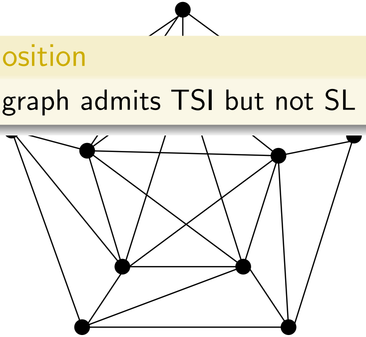
≠

embedded

Semilattice

≠

TSI



Proposition

This graph admits TSI but not SL

interval = path

$\neq$

chordal = skeletal = embedded skeletal tree

$\neq$

embedded tree

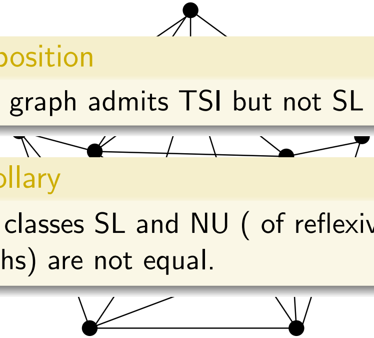
$\neq$

embedded

Semilattice

$\neq$

TSI

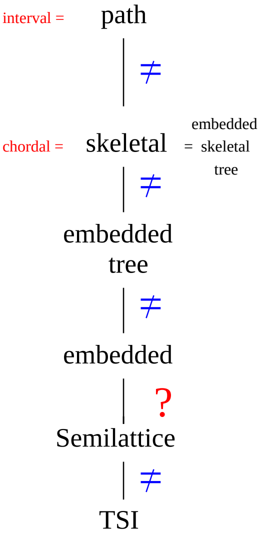


Proposition

This graph admits TSI but not SL

Corollary

The classes SL and NU (of reflexive graphs) are not equal.


interval =
path
≠
chordal =
skeletal
= embedded skeletal tree
≠
embedded tree
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embedded
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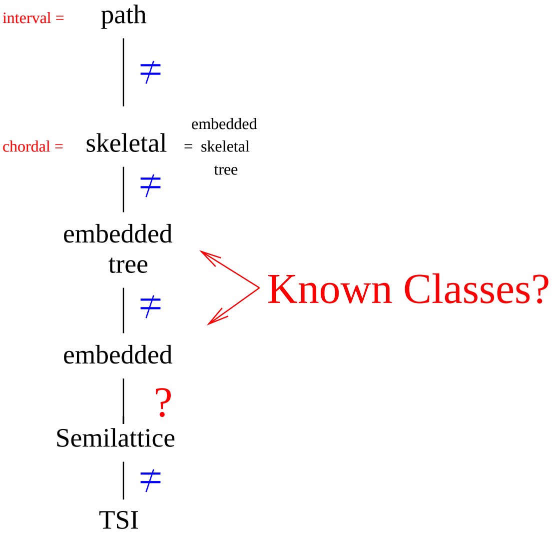
Semilattice
≠
TSI


interval =
path
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chordal =
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Semilattice
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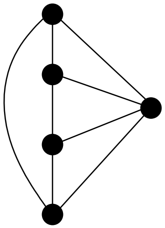
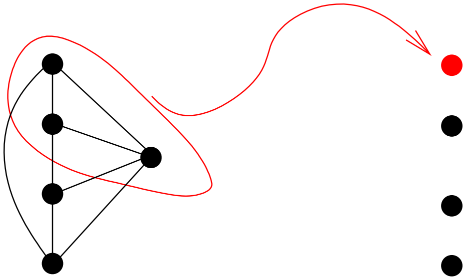
TSI
Known Classes?

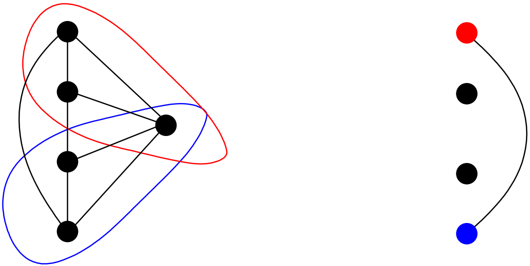
Chordal Reducible Graphs



Given a graph H,



Given a graph H, take its clique graph $\mathrm{CL}(H)$,



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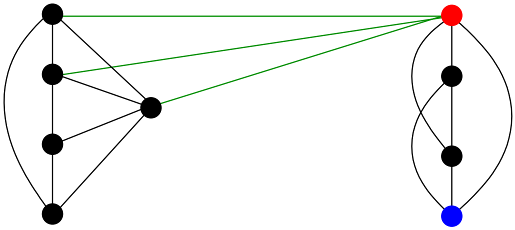
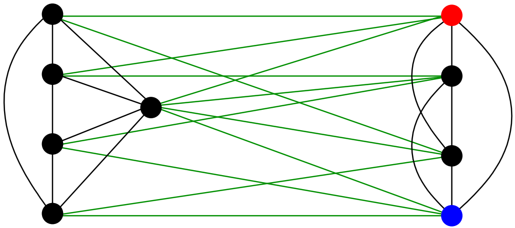


Given a graph H, take its clique graph $\mathrm{CL}(H)$, and add edges between them accoring to incidence: $\mathrm{CR}(H)$.



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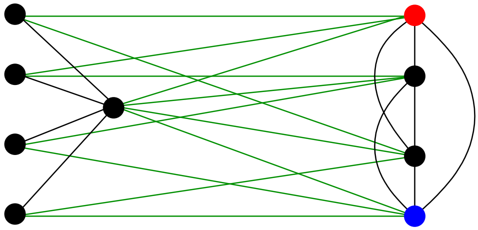
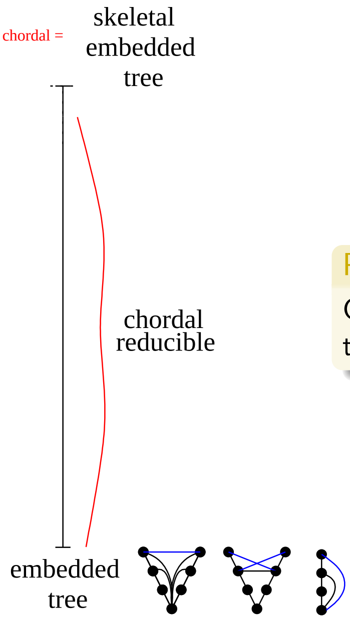


If we can remove edges from H such that it remains connected, and the full graph $\mathrm{CR}^*(H)$ is chordal, then H is chordal reducible .

- Chordal graphs are chordal reducible.
- Graphs with a universal vertex are chordal reducible.
- Chordal reducible graphs have NU of some arity.

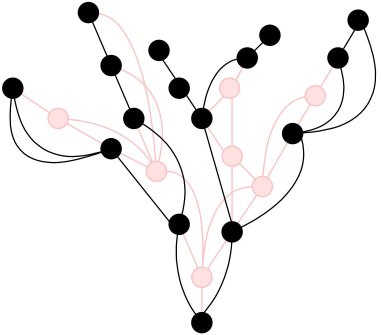
- Chordal graphs are chordal reducible.
- Graphs with a universal vertex are chordal reducible.
- Chordal reducible graphs have NU of some arity.
- Is there a poly time algorithm for recognising chordal reducible graphs?
- Are all graphs with 4-NU chordal reducible?
- Do chordal reducible graphs fit into our heirarchy?



Proposition

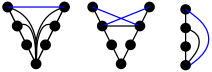
Chordal reducible graphs admit embedded tree polymorphisms.

chordal = skeletal embedded tree

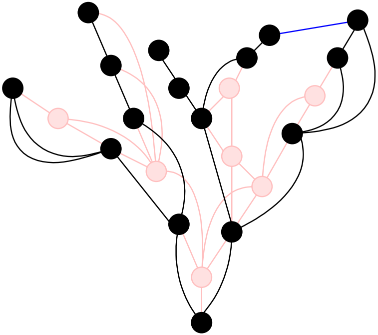


chordal reducible

embedded tree

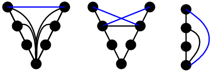


chordal = skeletal embedded tree



chordal reducible

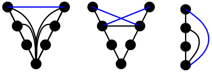
embedded tree

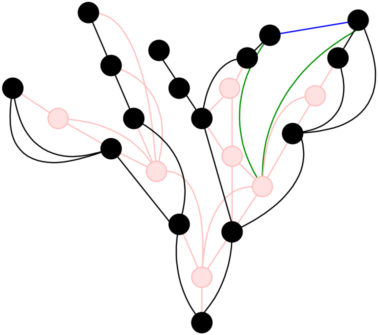


chordal = skeletal embedded tree

chordal reducible

embedded tree

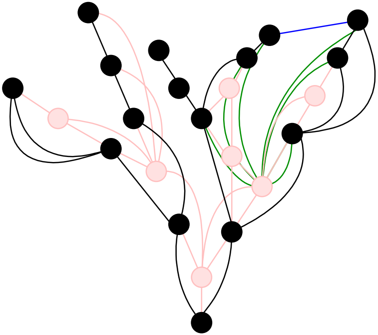


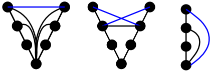


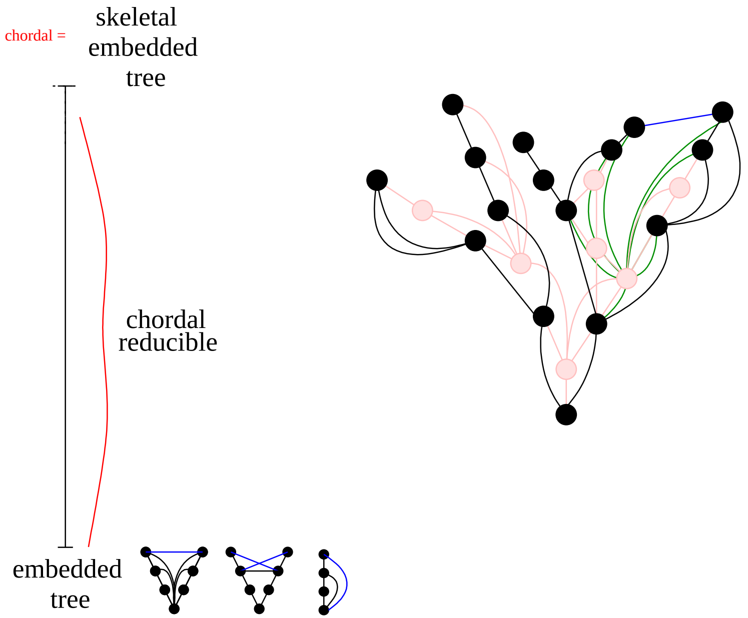
chordal = skeletal embedded tree

chordal reducible

embedded tree



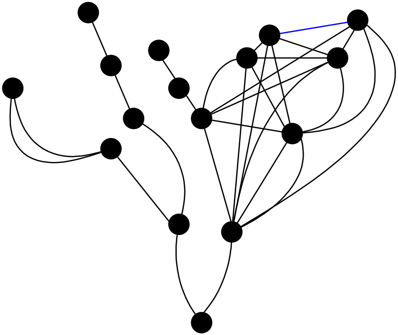


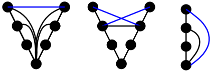

chordal =
skeletal embedded tree
chordal reducible
embedded tree

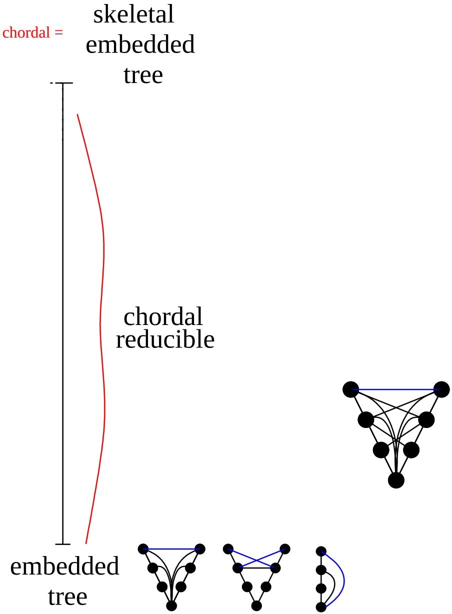
chordal = skeletal embedded tree

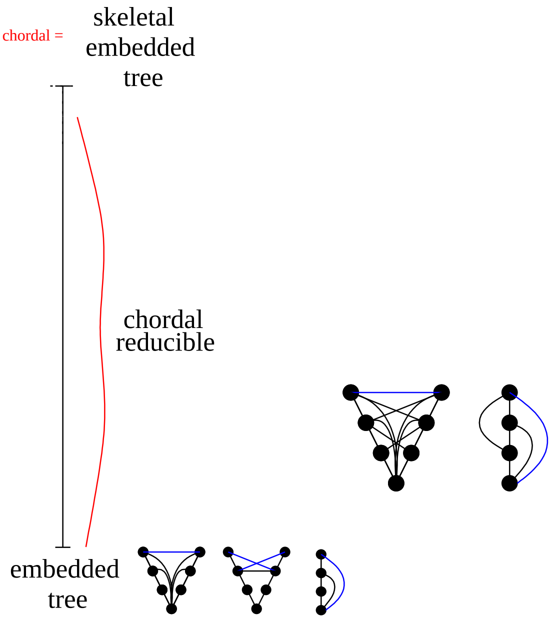
chordal reducible

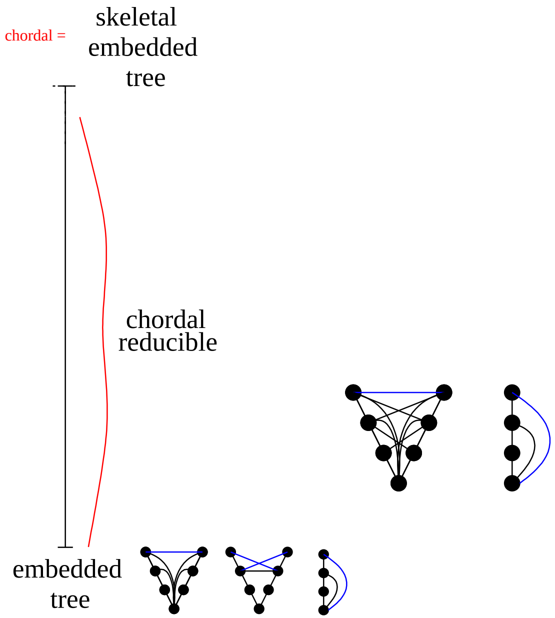
embedded tree

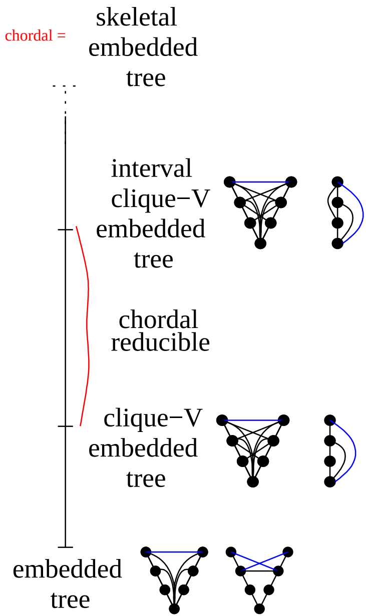




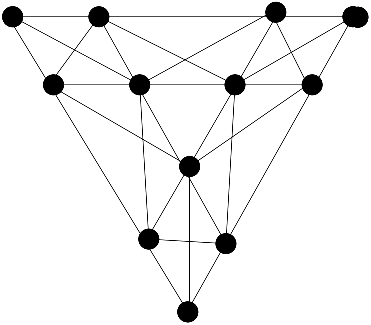

chordal =
skeletal embedded tree
chordal reducible
embedded tree


chordal =
skeletal embedded tree
chordal reducible
embedded tree


chordal =
skeletal embedded tree
chordal reducible
embedded tree


chordal =
skeletal embedded tree
interval clique−V embedded tree
chordal reducible
clique−V embedded tree
embedded tree

A Consequence



This graph has a 4-NU but no clique-V embedded tree polymorphism, so is not chordal reducible.

What we did

- Defined heirarchy of graph classes, generalising 'chordal' according to the type of SL polymorphism admitted.
- SL $\neq$ NU.
- 4-NU $\not\Rightarrow$ Chordal Reducible

Questions

- Does admitting a clique-V SL imply a graph is Chordal Reducible?
- Does 'SL' imply 'embedded SL'?
- Is there a poly-time algorithm for recognising graphs admitting
  - SL
  - clique-V SL
- Find a class of obstructions to SL that aren't obstructions to TSI.

Proof that NU $\neq$ SL

1. For a reflexive graph H let U_H be the structure defined
   - $V(U_H) = \mathrm{Powerset}(V(H))$
   - $(S, T) \in E(U_H)$ if for each $s \in S$ there is $t \in T$ with $(s, t) \in E(H)$, and vice versa.

   H has a *TSI* if and only if U_H retracts to copy of H induce by singleton vertices.
2. *NU* is preserved by retraction (NU is a variety).
3. U_H is in SL for any H: the semilattice $T < S$ if $S \subset T$ is polymorphic.

If H has a NU poly, then $H \in NU \setminus SL$ and we are done. Otherwise H has no NU poly. Since H has *TSI*, U_H retracts to H by (1), and so by (2) U_H has no NU poly. Thus $U_H \in SL \setminus NU$.