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REMARKS ON MCGAVRAN'S PAPER AND NISHIMURA'S RESULT

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This is the extended abstract for the informal seminor during the Buchstaber's lecture in KAIST, and summarizes the papre [K].

In the seminor, we focus on *oriented small covers* (i.e., its first Stiefel-Whitney class is zero; $w_1(M^n) = 0$) and *spin quasitoric manifolds* (i.e., $w_2(M^{2n}) = 0$).

Let $\Lambda(k) = (I_n \mid \Lambda'(k))$ be a *characteristic function* on an n-dimensional simple polytope P, where $k = \mathbb{Z}_2$ or $\mathbb{Z}$ (see [DJ91] or [BP02] for detail). By using the method of [NN05], we can easily show the following proposition for these manifolds.

Proposition 0.1. If $M(P,\ \Lambda(k))$ is an oriented n-dimensional small cover for $k = \mathbb{Z}_2$ or a spin $2n$-dimensional quasitoric manifold for $k = \mathbb{Z}$, then each row vector $(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)^t \in k^n$ in $\Lambda'(k)$ satisfies that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n \equiv 1 (\mathrm{mod}\ 2).$$

Therefore, if r is odd, the following 3-dimensional quasitoric manifold (Figure 1) has a spin structure (also see examples in [CMS]).

McGavran in [M76] studied 6-dimensional spin manifolds with T^3-actions. According his lemma (4.9 Lemma of [M76]), the above r in Figure 1 must be 0 or 1. However, if $|r| > 2$, the above example does not satisfy the McGavran's lemma.

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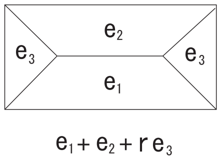

FIGURE 1

It follows that there is a mistake in McGavran's lemma (4.9 Lemma in [M76]). However, we can use his argument for small covers.

3-dimensional small covers have been studied by Izmestiev in [I01] for 3-colorable cases, Nakayama and Nishimura in [NN05] and [N04] for oriented cases (i.e., 4-colorable cases), and Lü and Yu in [LY] for all cases. In particular, Nishimura in [N04] shows the following theorem.

Theorem 0.2 (Nishimura)**.** *Let M be an oriented 3-dimensional small cover. Then M can be constructed from the real projective space $\mathbb{R}P(3)$ and the 3-dimensional torus T^3 by using finite times equivariant connected sums $\sharp$ and 2 equivariant surgeries $\natural$ and $\flat$.*

Here, $\sharp$, $\natural$ and $\flat$ are as in Figure 2.

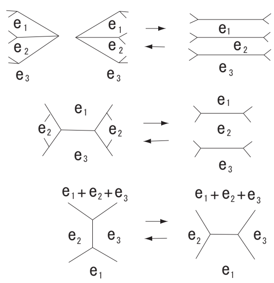

FIGURE 2. The first is the equivariant connected sum $\sharp$, the second is the equivariant surgery $\natural$, and the third is the another equivariant surgery $\flat$.

Using the McGavran's idea, we can improve Nishimura's Theorem 0.2 as the following main theorem.

Theorem 0.3. *Let M be an oriented 3-dimensional small cover. Then M can be constructed from the real projective space $\mathbb{R}P(3)$ and the 3-dimensional torus T^3 by using finite times equivariant connected sums $\sharp$ and equivariant surgeries $\natural$.*

In other words, the equivariant surgery $\flat$ can be constructed by $\sharp$ and $\natural$ as Figure 3.

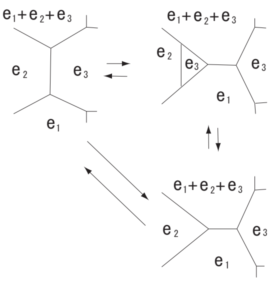

FIGURE 3. $\flat = (\sharp\Delta)^{-1} \circ \natural^{-1}$ and $\flat^{-1} = \natural \circ (\sharp\Delta)$.

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