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COHOMOLOGICAL RIGIDITY OF QUASITORIC MANIFOLDS AND SIMPLE CONVEX POLYTOPES

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An n-dimensional convex polytope is called *simple* if at each vertex exactly n facets (codimension one face) intersect. The notion of *quasitoric manifold* was first introduced by Davis and Januszkiewicz in [4] as a topological analogue of toric variety in algebraic geometry, which is a closed $2n$-dimensional manifold M^{2n} with a locally standard n-torus $T^n = (S^1)^n$ action such that its orbit space has a combinatorial structure of a simple convex polytope P^n. In this case we say M^{2n} is over P^n. Davis and Januszkiewicz named such manifold as *toric manifold*, but toric manifold is a well-established term for algebraic geometors as a nonsingular toric variety. So Buchstaber and Panov renamed it as quasitoric manifold in [1], and now this terminology is widely accepted among toric toopolgists.

One of the interesting questions on quasitoric manifold is the following cohomological rigidity question.

Question 0.1. *Let M and N be two quasitoric manifolds with isomorphic cohomology rings. Does this imply that M and N are homeomorphic?*

Masuda and Panov give a positive answer to the question for quasitoric manifolds whose cohomology rings are isomorphic to those of product of $\mathbb{C}P^1$'s in [5]. Namely, they show that any quasitoric manifold M^{2n} with $H^*(M^{2n}) \cong H^*(\prod \mathbb{C}P^1)$ is actually homeomorphic to $\prod \mathbb{C}P^1$. It is proved in two steps. First they show that any quasitoric manifold M^{2n} over a cube I^n with $H^*(M^{2n}) \cong H^*(\prod \mathbb{C}P^1)$ is homeomorphic to $\prod \mathbb{C}P^1$. Then they show that any quasitoric manifold M^{2n} with $H^*(M^{2n}) \cong H^*(\prod \mathbb{C}P^1)$ is actually over the cube I^n.

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The second step of their proof motivates the following definition of cohomological rigidity of simple convex polytope.

Definition 2. A simple convex polytope P is called *cohomologically rigid* (or simply rigid) if there exists a quasitoric manifold M over P and whenever there is another quasitoric manifold N over a simplex convex polytope Q with $H^*(M) \cong H^*(N)$ as graded rings, then P is combinatorially equivalent to Q.

Not all simple convex polytopes are rigid as the example in Section 4 of [6] shows. So the following question is reasonable to ask.

Question 0.3. *Which simple convex polytopes are rigid?*

By the previously mentioned work of [5], every cube are rigid.

In [3] Choi, Panov and Suh find some more rigid polytoes. Namely, the following theorem is the main result of [3]. A polytope is *triangle-free* if there is no triangular 2-dimensional face. For a simple convex polytope P and a vertex v of it, the vertex cut $\mathrm{vc}(P, v)$ is a polytope obtained from P by cutting out a simplex shaped neighborhood of v. If $\mathrm{vc}(P, v)$ is independent of a choice of a vertex v, then we write $\mathrm{vc}(P)$.

Theorem 4. [3] *The following polytopes are cohomologically rigid.*

(1) *Any polygon, i.e., any 2-dimensional simple convex polytope.*
(2) *Any triangle-free n-dimensional simple convex polytope with facet numbers $\leq 2n + 2$.*
(3) *Any product of simplices, i.e., $\prod \Delta^{n_i}$.*
(4) *Any vertex cut of a product of simplices, i.e.,* $\mathrm{vc}(\prod \Delta^{n_i})$.
(5) *Dodecahedron*

The theorem is proved using Tor-algebra and bigraded betti numbers $\beta^{-i,2j}$ of a polytope P, see Section 3.3 and 3.4 of [1] for definition of Tor-algebra and bigraded betti numbers.

In [2] it is proved that if M is a quasitoric manifold over a product of simpleces $\prod \Delta^{n_i}$ such that $H^*(M) \cong H^*(\prod \mathbb{C}P^{n_i})$, then $M \cong \prod \mathbb{C}P^{n_i}$ (homeomorphism). Since $\prod \Delta^{n_i}$ is rigid by Theorem 4, we can conclude that if M is a quasitoric manifold such that $H^*(M) \cong H^*(\prod \mathbb{C}P^{n_i})$ then $M \cong \prod \mathbb{C}P^{n_i}$ (homeomorphism), which generalizes the result in [5].

A simplicial complex of dimension $n - 1$ is called *Cohen-Macaulay* if there is a length n regular sequence in the face ring (or Stanley-Reisner ring) $k(K)$ where k is a field. Remember that the face ring of a simpicial complex with vertices $v_1, \ldots, v_m$ is the ring $k[v_1, \ldots, v_m]/I$ where I is the ideal generated by the set of monomials $v_{i_1} \cdots v_{i_k}$ where the vertices $v_{i_1}, \ldots, v_{i_k}$ does not form a simplex in K. Also remember that a sequence $\lambda_1, \ldots, \lambda_p$ of homogeneous elements in $k(K)$ is a *regular sequence* if it is algebraically independent and $k(K)$ is a free module over

$k(\lambda_1, \ldots, \lambda_p)$. It is known that for any simple convex n-polytope P, the dual $(\partial P)^*$ of its boundary is Cohen-Macaulay. The definition of cohomological rigidity of a simple convex polytope can be generalized to that of a Cohen-Macaulay complex as follows.

Definition 5. An $(n-1)$-dimensional Cohen-Macaulay complex K is rigid if for any $(n-1)$-dimensional Cohen-Macaulay complex K' and for ideals $J \subset k(K)$ and $J' \subset k(K')$ generated by degree 2 regular sequences of length n, $k(K)/J \cong k(K')/J'$ implies $k(K) \cong k(K')$.

So far no example of rigid Cohen-Macaulay complex which is not a dual of the boundary of a simple convex polytope is known.

For an n-dimensional simple convex polytope P^n with m facets, David and Januszkiewicz constructed in [4] a T^m-manifold $\mathcal{Z}_P$ with orbit space P^n such that for any quasitoric manifold $\pi : M^{2n} \to P^n$ there is a principal T^{m-n}-bundle $\mathcal{Z}_P \to M$ whose composite map with π is the orbit map $\rho : \mathcal{Z}_P \to P$. This manifold is called the *moment angle complex* (or manifold) of P. Moment angle complex can be defined for arbitrary simplicial complex K. It is prove by Buchstaber and Panov that the cohomology $H^*(\mathcal{Z}_K)$ of $\mathcal{Z}_K$ for a Cohen-Macaulay complex is isomorphic to the Tor-algebra of K, see Theorem 7.13 of [1]. Buchstaber modified Question 0.3 in terms of moment angle complex of a simplicial complex.

Question 0.6. *Let K and K' be two simplicial complexes such that $H^*(\mathcal{Z}_k : k) \cong H^*(\mathcal{Z}_{k'} : k)$ as bigraded k-algebra. When does this imply a combinatorial equivalence $K \approx K'$?*

It is proved in [3] that if M (resp. N) is a quasitoric manifold over P (resp. Q) such that $H^*(M) \cong H^*(N)$ as graded rings, then $H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_Q)$. So if P is cohomologically rigid, then it is also rigid in the sense of Question 0.6, namely if $K = (\partial P)^*$ and K' is another simplicial complex such that $H^*(\mathcal{Z}_K : k) \cong H^*(\mathcal{Z}_{K'} : k)$ as bigraded k-algebra, then $K \approx K'$. Note that the moment angle complex $\mathcal{Z}_K$ is completely determined by the combinatorial structure of K. So Question 0.6 is purely combinatorial question about K. On the other hand Question 0.3 is related with quasitoric manifolds over a polytope. However existence of quasitoric manifold over P is completely determined by the combinatorial structure of P. So it might be possible that two Questions 0.3 and 0.6 are equivalent, but at this moment it is not clear.

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