1 Introduction

Outline

- The definition of the Teichmüller space of 2-orbifolds
- The geometric cutting and pasting and the deformation spaces
- The decomposition of 2-orbifolds into elementary orbifolds.
- The Teichmüller spaces of 2-orbifolds

Some helpful references

- S. Choi, Geometric structures on orbifolds and holonomy representations, Geometriae Dedicata 104: 161 – 199, 2004.
- S. Choi and W. Goldman, The deformation spaces of convex RP^2-structures on 2-orbifolds, American Journal of Mathematics 127, 1019–1102 (2005)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- K. Ohshika, Teichmüller spaces of Seifert fibered manifolds with infinite π_1. Topology Appl. 27 (1987), no. 1, 75 – 93. (Surikaisekikenkyusho Kukyuroku No. 542 (1985), 119 – 127.)
- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977.

2 The definition of the Teichmüller space of 2-orbifolds

Definition of Teichmüller spaces of 2-orbifolds

- A *hyperbolic* structure on a 2-orbifold is a geometric structure modeled on H^2 with the isometry group $\mathrm{PSL}(2, \mathbb{R})$.
- The *Teichmüller* space $\mathcal{T}(M)$ of a 2-orbifold M is the deformation space of hyperbolic structures on the 2-orbifold.
- As before, we reinterpret the space as
  - The set of diffeomorphisms $f : M \to M'$ for M an orbifold and M' a hyperbolic 2-orbifold.
  - The equivalence relation $f : M \to M'$ and $g : M \to M$" if exists a hyperbolic isometry $h : M' \to M$" so that $h \circ f$ is isotopic to g.
  - The quotient space is same as above.

- A necessary condition for an orbifold to have a hyperbolic structure is that the orbifold euler characteristic be negative: This follows from the Gauss-Bonnet theorem. Here the negative of the hyperbolic area is the Euler characteristic times 2π.
- A closed 2-orbifold with a complex structure has a unique hyperbolic structure provided it is compact and has negative Euler characteristic.
- The deformation space of complex structures on a closed 2-orbifold is identical with the Teichmuller space as defined here by the uniformization theorem.

3 The geometric cutting and pasting and the deformation spaces

The geometric cutting and pasting and the deformation spaces

- A compact geodesic 1-orbifold without boundary points in the interior of a 2-orbifold Σ are either
  - a closed geodesic in the interior or entirely in the boundary of $|\Sigma|$ or
  - a segment with two silvered points which are either at silvered edges or cone-points of order two. The topological interior is either in the interior or entirely in the boundary of $|\Sigma|$.
- The geometric type is classified by length and the topological type. Such a geodesic 1-orbifold is covered by a closed geodesic in some cover of the 2-orbifold, which is a surface.
- The *Teichmüller space* $\mathcal{T}(I)$ for a 1-orbifold I is the product of the space of lengths $\mathbb{R}^+$s for each component of I.
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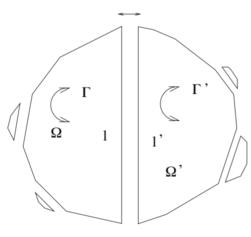
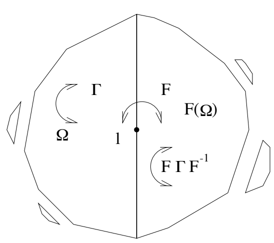
Geometric constructions.

- Recall the type of topological constructions with 1-orbifolds. Suppose they are boundary components of 2-orbifolds whose components have negative Euler characteristics.
  - (A)(I) Pasting or crosscapping along simple closed curves.
  - (A)(II) Silvering or folding along a simple closed curve.
  - (B)(I) Pasting along two full 1-orbifolds.
  - (B)(II) Silvering or folding along a full 1-orbifold.
- Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric version of the above.

Geometric constructions.

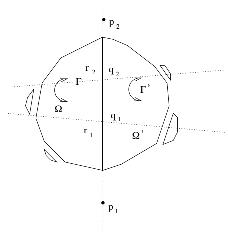
- Suppose that the involved 1-orbifolds are geodesic boundary components of a hyperbolic 2-orbifolds.
  - (A)(I). For pasting two closed geodesics, we have a $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. (Here the length of two closed geodsics have to be the same.)
  - (A)(I) For cross-capping, we have a unique isometry. The isometry has to be a slide reflection of distance half the length of the closed geodesic. (There is no conditions.)
  - 

Geometric constructions.

- (A)(II). For folding a closed geodesics, we have a $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The choice depends on the choice of two fixed points of the pasting map. The distance is the half of length of the closed geodesic. (no condition)
- (A)(II) For silvering, we have unique isometry to do this. (no condition)
- 

Geometric constructions.

- (B)(I). For pasting along two geodesic full 1-orbifolds, We have a unique way to do this. The lengths of the orbifolds have to be the same.
- (B)(II) For silvering and folding, we have unique isometry to do this. (no condition)

- 

Teichmuller spaces under the geometric operations

(A)(I)(1) Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \Delta \subset \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration, where Δ is the subset of $\mathcal{T}(\Sigma')$ where b and b' have equal legnths.

(A)(I)(2) Let Σ'' be obtained from Σ' by cross-capping. The resulting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a diffeomorphism.

(A)(II)(1) Let Σ'' be obtained from Σ' by silvering. The clarifying map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a diffeomorphism.

(A)(II)(2) Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration.

(B)(I) Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \Delta \subset \mathcal{T}(\Sigma')$$

is a diffeomorphism where Δ is a subset of $\mathcal{T}(\Sigma')$ where the lengths of b and b' are equal.

(B)(II) Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a diffeomorphism.

4 The decomposition of 2-orbifolds into elementary orbifolds.

Topological decomposition of hyperbolic 2-orbifolds into elementary orbifolds along geodesic 1-orbifolds.

- Suppose that Σ is a compact hyperbolic orbifold with $\chi(\Sigma) < 0$ and geodesic boundary.
- Let $c_1, \ldots, c_n$ be a mutually disjoint collection of simple closed curves or 1-orbifolds so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \cdots - c_n$ is negative.
- Then $c_1, \ldots, c_n$ are isotopic to simple closed geodesics or geodesic full 1-orbifolds $d_1, \ldots, d_n$ respectively where $d_1, \ldots, d_n$ are mutually disjoint.

Elementary 2-orbifolds.
We require the boundary components be geodesics.

(P1) A pair-of-pants.

(P2) An annulus with one cone-point of order n. ($A(; n)$)

(P3) A disk with two cone-points of order p, q, one of which is greater than 2. ($D(; p, q)$)

(P4) A sphere with three cone-points of order p, q, r where $1/p + 1/q + 1/r < 1$. ($\mathbf{S}^2(; p, q, r)$)

(A1) An annulus with one boundary component a union of a singular segment and one boundary-orbifold. (2-*pronged crown* and $A(2, 2;)$.) It has two corner-reflectors of order 2 if the boundary components are silvered.

(A2) An annulus with one boundary component of the underlying space in a singular locus with one corner-reflector of order n, $n \geq 2$. (The other boundary component is a closed geodesic which is the boundary of the orbifold.) (We call it a *one-pronged crown* and denote it $A(n;)$.)

(A3) A disk with one singular segment and one boundary 1-orbifold and a cone-point of order greater than or equal to three ($D^2(2, 2; n)$).

(A4) A disk with one corner-reflector of order m and one cone-point of order n so that $1/2m + 1/n < 1/2$ (with no boundary orbifold). ($n \geq 3$ necessarily.) ($D^2(m; n)$.)

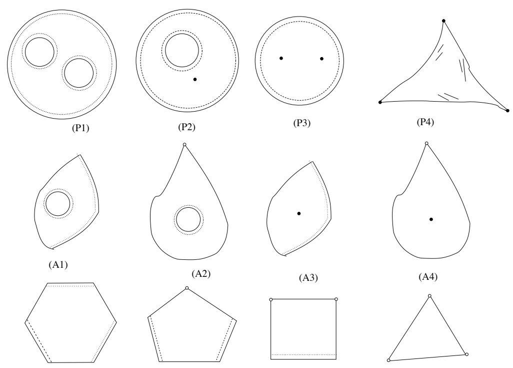
(D1) A disk with three edges and three boundary 1-orbifolds. No two boundary 1-orbifolds are adjacent. (We call it a *hexagon* or $D^2(2, 2, 2, 2, 2, 2;)$.)

(D2) A disk with three edges and two boundary 1-orbifolds on the boundary of the underlying space. Two boundary 1-orbifolds are not adjacent, and two edges meet in a corner-reflector of order n, and the remaining one a segment. (We called it a *pentagon* and denote it by $D^2(2, 2, 2, 2, n;)$.)

(D3) A disk with two corner-reflectors of order p, q, one of which is greater than or equal to 3, and one boundary 1-orbifold. The singular locus of the disk is a union of three edges and two corner-reflectors. (We call it a *quadrilateral* or $D^2(2, 2, p, q;)$.)

(D4) A disk with three corner-reflectors of order p, q, r where $1/p+1/q+1/r < 1$ and three edges (with no boundary orbifold). (We call it a *triangle* or $D^2(p, q, r;)$.)

The diagram for elementary orbifolds

- 

  The elementary orbifolds. Arcs with dotted arcs next to them indicate boundary components. Black points indicate cone-points and white points the corner-reflectors.

The geometric decomposition into elementary orbifolds

- Let Σ be a compact hyperbolic orbifold with $\chi(\Sigma) < 0$ and geodesic boundary.
- Then there exists a mutually disjoint collection of simple closed geodesics and mirror- or cone- or mixed-type geodesic 1-orbifolds so that Σ decomposes along their union to a union of elementary 2-orbifolds or such elementary 2-orbifolds with some boundary 1-orbifolds silvered additionally.

5 The Teichmüller spaces for 2-orbifolds

Thurston's theorem

- Let Σ be a compact 2-orbifold with empty boundary and negative Euler characteristic diffeomorphic to an elementary 2-orbifold.
- Then the deformation space $\mathcal{T}(\Sigma)$ of hyperbolic $\mathbb{RP}^2$-structures on Σ is homeomorphic to a cell of dimension $-3\chi(X_\Sigma) + 2k + l + 2n$ where X_Σ is the underlying space and k is the number of cone-points, l is the number of corner-reflectors, and n is the number of boundary full 1-orbifolds.

Strategy of proof

- Proposition A: for each elementary 2-orbifold S, $\mathcal{T}(S)$ is homeomorphic to $\mathcal{T}(\partial S)$, where $\mathcal{T}(\partial S)$ is the product of $\mathbb{R}^+$ for each component of ∂S corresponding to the hyperbolic-metric lengths of components of ∂S.

- Then for hyperbolic structures, to obtain a bigger orbifold, we need to use the above result about the Teichmüller spaces under geometric decompositions.

The generalized hyperbolic triangle theorem

- A generalized triangle in the hyperbolic plane is one of following:

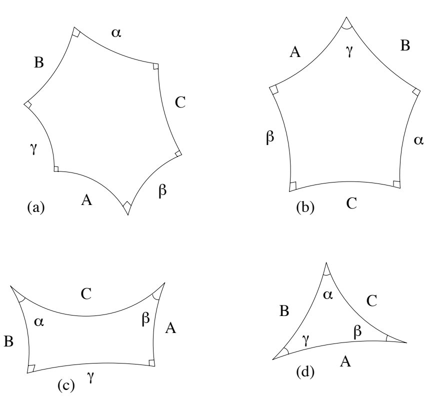
  (a) A hexagon: a disk bounded by six geodesic sides meeting in right angles labeled $A, \beta, C, \alpha, B, \gamma$.

  (b) A pentagon: a disk bounded by five geodesic sides labeled A, B, α, C, β where A and B meet in an angle γ, and the rest of the angles are right angles.

  (c) A quadrilateral: a disk bounded by four geodesic sides labeled A, C, B, γ where A and C meet in an angle β, C and B meet in an angle α and the two remaining angles are right angles.

  (d) A triangle: a disk bounded by three geodesic sides labeled A, B, C where A and B meet in an angle γ and B and C meet in an angle α and C and A meet in angle β.

The generalized hyperbolic triangles

- 

The trigonometry

- For generalized triangles in the hyperbolic plane,

$$
\begin{aligned}
\text{(a)} \quad \cosh C &= \frac{\cosh\alpha\cosh\beta + \cosh\gamma}{\sinh\alpha\sinh\beta} \\
\text{(b)} \quad \cosh C &= \frac{\cosh\alpha\cosh\beta + \cos\gamma}{\sinh\alpha\sinh\beta} \\
\text{(c)} \quad \sinh A &= \frac{\cosh\gamma\cos\beta + \cos\alpha}{\sinh\beta\sin\gamma} \\
\text{(d)} \quad \cosh C &= \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta} \qquad (1)
\end{aligned}
$$

- In (a), (α, β, γ) can be any positive numbers.
- In (b), (α, β) can be any positive numbers and γ in $(0, \pi)$
- In (c), (α, β) can be any positive real numbers in $(0, \pi)$ satisfying $\alpha + \beta < \pi$, and γ any real number.
- In (d), (α, β, γ) can be any real numbers in $(0, \pi)$ satisfying $\alpha + \beta + \gamma < \pi$.

The proof of Proposition A.

- The following lemma implies Proposition A for elementary 2-orbifolds of type (D1), (D2), (D3), and (D4).
- Silvered edges labeled by the capital letters A, B, C. Assign to each vertex an angle of the form π/n (where $(n > 1$ is an integer), for which it is a corner-reflector of that angle. Each edge labeled by Greek letters α, β, γ is a boundary full 1-orbifold.
- Then in cases (a), (b), (c), (d) $\mathcal{F} : \mathcal{T}(P) \to \mathcal{T}(\partial P)$ for each of the above orbifolds P is a homeomorphism; that is, $\mathcal{T}(P)$ is homeomorphic to a cell of dimension 3, 2, 1, or 0 respectively.
- Let S be an elementary 2-orbifold of type (A1), (A2), (A3), or (A4).
- Then $\mathcal{F} : \mathcal{T}(S) \to \mathcal{T}(\partial S)$ is a homeomorphism. Thus, $\mathcal{T}(S)$ is a cell of dimension $2, 1, 1$, or 0 when S is of type (A1), (A2), (A3) or (A4) respectively. In case (A4), $\mathcal{T}(S)$ is a single point.
- For elementary orbifolds of type (P1),(P2),(P3), or (P4), we simply notices that they double covers orbifolds of type (D1),(D2),(D3), or (D4) which is realized as isometries where each of the boundary components do the same. In fact, the isometry can be explictly constructed by taking shortest geodesics between boundary components.

The steps to prove Theorem A.

- Let a 2-orbifold Σ, each component of which has negative Euler characteristic, be in a class $\mathcal{P}$ if the following hold:

  (i) The deformation space of hyperbolic $\mathbb{RP}^2$-structures $\mathcal{T}(\Sigma)$ is diffeomorphic to a cell of dimension

  $$-3\chi(X_\Sigma) + 2k + l + 2n$$

  where k is the number of cone-points, l the number of corner-reflectors, n is the number of boundary full 1-orbifolds.

  (ii) There exists a principal fibration

  $$\mathcal{F} : \mathcal{T}(\Sigma) \to \mathcal{T}(\partial\Sigma)$$

  with the action by a cell of dimension $\dim \mathcal{T}(\Sigma) - \dim \mathcal{T}(\partial\Sigma)$.

- Let Σ be a 2-orbifold whose components are orbifolds of negative Euler characteristic, and it splits into an orbifold Σ' in $\mathcal{P}$.

- We suppose that (i) and (ii) hold for Σ', and show that (i) and (ii) hold for Σ. Since Σ eventually decomposes into a union of elementary 2-orbifolds where (i) and (ii) hold, we would have completed the proof.

- The proof follows by going through each of the constructions....

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