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Preface

One aim of mathematics is to explore many objects purely defined and created out of imaginations in the hope that they will explain many unknown and unsolved phenomenons in mathematics. Manifolds were studied much in 20th century. Perhaps, we mathematicians develop from here to more abstract theories that can accommodate many things that we promised to solve in the earlier part of 20th century. Orbifolds might be a small step away in the right directions as orbifolds have all the notions of manifold theory easily generalized as discovered by Satake.

The theory of orbifolds is a natural generalization of the notion of manifolds. Orbifolds can be more useful tool than manifolds in many ways involving in classification of knots, graphs imbeddings, theoretical physics and so on. At least in two or three-dimensions, orbifolds are easy to produce and classifiable using Thurston's geometrization program. However, in higher dimensions, these topics are still very mysterious where many mathematical and theoretical physicists are working.

The covering space theory is explained using both the fiber-product approach of Thurston and path-approach by Haefliger. The main part of the book is the geometric structures on orbifolds. We define the deformation space of geometric structures on orbifolds and prove the local homeomorphism theorem that the deformation spaces are locally homeomorphic to the representation spaces of the fundamental groups. The main emphasis are on studying geometric structures and ways to cut and paste the geometric structures. These form a main topic of this book and will hopefully aid the reader in studying many possible geometric structures on orbifolds including affine, projective, and so on. Also, these other types of geometries seems to be of use in Mirror symmetry and so on.

In this book, we tried to collect the theory of orbifold scattered in various literatures. Here we set out to write down the traditional approach to orbifolds using charts and include the categorical definition using groupoids and compare them. We think that understanding both theory necessary.

This book is intentionally made to be short as there are many extensive writings on the subject already available. Instead of writing every proofs down, we try to explain the reasonings behind the proof and pointing to where the proofs can be found. This was done in the hope that the readers can follow the reasonings without having to understand the full details of the proof, and can fast-track into this field. Also, the book is also hopefully self-sufficient for people who do not wish to delve into technical details.

This book is based on a course the author gave in the fall term of 2008 at Tokyo Institute of Technology. I thank very much the hospitality of the Department of Mathematical and Computing Sciences.

The book was intended for the advanced undergraduates and the beginning graduate students who understand some differentiable manifold theory, Riemannian geometry, some manifold topology, algebraic topology and Lie group actions. But we do include sketches of these theories in the beginning of the book as a review. Unfortunately, some familiarity with category theory is needed where the author cannot provide a sufficiently good introduction.

Suhyoung Choi

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Chapter 1

The introduction

1.1 Introduction

This course will introduce 2-orbifolds and geometric structures on them. It will be for senior or master level course. Some background in topology, manifold theory, differential geometry, and particularly category theory would be helpful.

As one knows, the manifold theory enjoyed a great deal of attention in 20th century mathematics enticing many talented mathematicians. Orbifolds are natural generalizations of the notion of manifolds introduced by Satake. They provide a natural setting to study discrete group actions on manifolds. In fact, orbifolds have most notions developed from manifold theory carried over to them although perhaps in an indirect manner, using the language of category theory. Indeed, to make the orbifold theory most rigorously understood, only the category theory provide the most natural setting.

Also, it is thought that orbifolds are integral part of theoretical physics such as string theory, and they have natural generalizations in algebraic geometry. We also think that orbifold theory has important role in understanding knots, links, and graph imbeddings.

For 2-manifolds, it was known from the classical time that classical geometry provide a sharp insight into the topology of surfaces and their groups of automorphisms as observed by Dehn and others. In later 1970s, Thurston proposed a program to generalize these kinds of insights to 3-manifold theory. This program is now completed by the proof of the Geometrization conjecture as is well-known. The computer programs as initiated by Thurston and completed mainly by Weeks, Hodgson, and so on, now compute most topological properties of 3-manifolds completely given the 3-manifold topological data.

It seems that the direction of the research in low-dimensional manifold theory currently is perhaps to complete the understanding of 3-manifolds by volume ordering, arithmetic properties, and group theoretical properties. Perhaps, we should start to move to higher-dimensional manifolds and to more applied areas.

One area which can be of possible interest to is to study the projectively flat, affinely flat, or conformally flat structures on 3-manifolds. This will complete the understanding of all classical geometric properties of 3-manifolds. This aspect is related to understanding all representations of the fundamental groups of 3-manifolds into Lie groups where many interesting questions still remain, upon which we mention that we are yet to understand fully the 2-orbifold or surface fundamental group representations into Lie groups.

We will learn the 2-dimensional orbifold theory and the geometric structures on them. We will cover some of the background materials such as Lie group theory, principal bundles and connections. The theory of orbifolds has much to do with discrete subgroups of Lie groups but has more topological flavors. We discuss the topology of 2-orbifolds including covering spaces and orbifold-fundamental groups. The fundamental groups of 2-orbifolds include many interesting infinite groups. We obtain the understanding of the deformation space of hyperbolic structures on a 2-orbifold, which is the space of discrete $\mathrm{PSL}(2,\mathbb{R})$-representations of the 2-orbifold fundamental group equivalent up to conjugations. Finally, we will survey the deformation spaces of projective structures on 2-orbifolds, which corresponds to the spaces of $\mathrm{PGL}(3,\mathbb{R})$-representations of the fundamental groups.

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This book has three parts. In Part I consisting of Chapters 1 and 2, we review manifold theory with G-structures. In Part II, consisting of Chapters 3,4, and 5, we present the topological theory of orbifolds. In Part III, consisting of Chapters 6,7, and 8, we present the theory of geometric structures of orbifolds.

In Chapter 2, Manifolds and differentiable structures, we will review smooth structures on manifolds and orbifolds starting from topological constructions, homotopy groups and covering spaces, simplicial manifolds including examples of surfaces. Then we move onto pseudo-groups and G-structures. Next, we review Lie groups and principal bundle theory in terms of smooth manifold theory. Finally, we interpret the G-structures in principal bundle theory.

In Chapter 3, Lie groups and geometry, we first review the Lie group theory in more detail. Then we review Euclidean, spherical, affine projective, and conformal geometry reviewing their properties and Lie groups acting as symmetries on them. Next, we go over to hyperbolic geometry. We begin from Lorentzian hyperboloid model and move onto Beltrami-Klein model, conformal model and the upper half-space model. Hyperbolic triangle laws will be studied also and the isometry groups of hyperbolic spaces will be introduced. We will also discuss the discrete group actions on manifolds using the Poincare fundamental polyhedron theorem and discuss Coxeter groups, triangle groups, and crystallographic groups.

In Chapter 4, Topology of orbifolds, compact group actions, we will start reviewing compact group actions on manifolds. We will talk about the orbit spaces and tubes, smooth actions and equivariant triangulations. Next, we will introduce orbifolds from the classical definition by Satake using atlas of charts. We define singular sets and suborbifolds. We will also present orbifolds as Lie groupoids from category theory as was initiated by Haefliger. We will present differentiable structures on orbifolds, bundles over orbifolds, and Gauss-Bonnet theorem for orbifolds. To find the universal covers of 2-orbifolds, we start from defining covering spaces of orbifolds and discuss how to obtain a fiber-product of two covering orbifolds. This lead us to the universal covering orbifolds and deck transformation groups and their properties such as uniqueness. We will also present the path-approach to the universal covering orbifolds as initiated by Haefliger.

In Chapter 5, Topological constructions of 2-orbifolds, we will present how to compute the Euler charateristic of 2-orbifolds including Riemann-Hurwitz formula. We will show how to topologically construct 2-orbifolds from other 2-orbifolds using cutting and sewing methods. This will be reinterpretted in two other manner. We also discuss good and bad orbifolds and the classification of 2-orbifolds.

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In Chapter 6, Geometric structures on orbifolds, we will define geometric structures on orbifold using atlas of chart methods, developing map and holonomy homomorphism methods, and as a cross-section to a bundle. We will show that these definitions are equivalent. We will also show that orbifolds admitting a geometric structure is always good, that is covered by a manifold. Here, we will define the deformation spaces of geometric structures on 2-orbifolds and indicate a short proof of local homeomorphism from the deformation space to the space of representations of the fundamental group to the Lie group G.

In Chapter 7, The deformation spaces of hyperbolic structures on 2-orbifolds, i.e., the Teichmuller space, we will first define the Teichmuller space and present geometric cutting and pasting constructions of hyperbolic structures on 2-orbifolds. We show that any 2-orbifolds decompose into elementary orbifolds. We will show how to compute the Teichmuller spaces of elementary orbifolds using hyperbolic trigonometry and piece these together to understand the Teichmuller space of the 2-orbifold. Finally, we study the deformation spaces of real projective structures on 2-orbifolds. We will use the method very similar to the above chapter. We decompose 2-orbifolds into elementary 2-orbifolds and determine the deformation space there and reassemble.

Our principal source for this lecture note is [Thurston (10)]. However, we do not go into his generalization of Andreev theorem. Also, [Thurston (11)] is a good source of many materials here.

Some standard text giving us preliminary viewpoint and alternative viewpoints of the foundational materials for this paper are many. [**(author?)** (Kobayashi and Nomizu)] provides us a good introductions to connections on principal bundles and [**?**)] and [**(author?)** (Ivey and Landsberg)] give us more differential geometric view point of geometric structures. [**(author?)** (Bredon)] is a good source for understanding the local orbifold group actions. Finally, [**(author?)** (Berger)] provides us with the knowledge of geometry that are probably most prevalent in this book.

Chapter 2

Manifolds and differentiable structures

2.1 Introduction

In this chapter, we will review many notions in manifold theory that generalize to orbifold theory.

We begin by reviewing manifold and simplicial manifolds beginning with cell-complexes, homotopy and covering theory. The following theories for manifolds will be transfered to the orbifolds. We will briefly mention them here as a "review" and will develop them for orbifolds later (mostly for 2-dimensional orbifolds). We follow coordinate-free approach to differential geometry. We do not need to actually compute curvatures and so on.

- Lie groups and group actions
- Pseudo-groups and $\mathcal{G}$-structures
- Differential geometry: Riemanian manifolds.
- Principal bundles and connections, flat connections

Some of these are standard materials in differentiable manifold course and in an advanced differential geometry course. We will not give proofs in Chapters 2 and 3 but will indicate one when necessary.

2.1.1 *Manifolds*

The useful methods of topology comes from taking equivalence classes and finding quotient topology. Given a topological space X with an equivalence relation, the quotient topology on $X/\sim$ is the topology so that for any function $f : X \to Y$ inducing a well-defined function $f' : X/\sim \to Y$, f' is continuous if and only if f is continuous. This translates to the fact that a subset U of $X/\sim$ is open if and only if $p^{-1}(U)$ is open in X for the quotient map $p : X \to X/\sim$.

A *cell* is a topological space homeomorphic to an n-dimensional ball defined in $\mathbb{R}^n$.

We will mostly use cell-complexes: (See [**(author?)** (Hatcher)] pages 5–7.) A *cell-complex* is a topological space which is a union of n-skeletons which is defined inductively. A 0-skeleton is a discrete set of points. A $n+1$-skeleton X^{n+1} is obtain from n-skeleton X^n as a quotient space of $X^n \cup \bigcup_{\alpha \in I} D^{n+1}_\alpha$ for a collection of $(n+1)$-dimensional balls D^{n+1}_α for $\alpha \in I$ with a collection of functions $f_\alpha : \partial D^{n+1}_\alpha \to X^n$ so that the equivalence relation is given by $x \sim f_\alpha(x)$ for $x \in \partial D^{n+1}_\alpha$. To obtain the topology of X, we use the weak topology that a subset U of X is open if and only if $U \cap X^n$ is open for every n. Most of the times, cell-complexes will be a *finite* ones, i.e., has finitely many cells.

A topological n-dimensional manifold (n-manifold) is a Hausdorff space with countable basis and charts to Euclidean spaces E^n; e.g curves, surfaces, 3-manifolds. The charts could also go to a positive half-space H^n. Then the set of points mapping to R^{n-1} under charts is well-defined is said to be the boundary of the manifold. By the invariance of domain theorem, we see that this is a well-defined notion.

For example, $\mathbb{R}^n$ and H^n themselves or open subsets of $\mathbb{R}^n$ or H^n are manifolds of dimension n.

The unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ is a standard example. $\mathbb{R}P^n$ the real projective space.

An n-ball is a manifold with boundary. The boundary is the unit sphere $\mathbf{S}^{n-1}$.

Given two manifolds M_1 and M_2 of dimensions m and n respectively. The product space $M_1 \times M_2$ is a manifold of dimension $m+n$.

An annulus is a disk removed with the interior of a smaller disk. It is also homeomorphic to a circle times a closed interval.

Example 2.1.1. The n-dimensional torus T^n is homeomorphic to the product of n circles $\mathbf{S}^1$. (For 2-torus, see `http://en.wikipedia.org/wiki/Torus` for its imbeddings in $\mathbb{R}^3$ and so on.)

Recall that a group G acts on a manifold M, if there is a differentiable map $k : G \times M \to M$ so that $k(g, k(h, e)) = k(gh, e)$ and $k(e, x) = x$ for the identity $e \in G$. Given an action, there is a homomorphism $G \to Diffeo(M)$ so that an element $g \in G$ goes to a diffeomorphism g' sending x to $k(g, x)$ where $Diffeo(M)$ is the group of diffeomorphisms of M.

Given a group G acting on a manifold M, the quotient space $M/\sim$

where $\sim$ is given by $x \sim y$ if and only if $x = g(y), g \in G$ is denoted by M/G. Let T_n be a group of translations generated by $T_i : x \mapsto x + e_i$ for each $i = 1, 2, .., n$. Then $\mathbb{R}^n/T_n$ is homeomorphic to T^n.

Example 2.1.2. We define the *connected sum* of two n-manifolds M_1 and M_2. Remove the interiors of two tamely imbedded closed balls from M_i for each i. Then each M_i has a boundary component homeomorphic to $\mathbf{S}^{n-1}$. We identify the spheres. Take many 2-dimensional tori or projective plane and do connected sums. Also remove the interiors of some disks. We can obtain all compact surfaces in this way. `http://en.wikipedia.org/wiki/Surface`

2.1.2 *Some homotopy theory*

Let X and Y be topological spaes. A *homotopy* is a map $F : X \times I \to Y$ for an interval I. Two maps f and $g : X \to Y$ are *homotopic* by a homotopy F if $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all x. The homotopic property is an equivalence relation on the set of maps $X \to Y$. A *homotopy equivalence* of two spaces X and Y is a map $f : X \to Y$ with a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to I_X and I_Y respectively. (See Hatcher [**(author?)** (Hatcher)] for details of homotopy theory presented here.)

The fundamental group of a topological space X is defined as follows: A *path* is a map $f : [a, b] \to X$ for an interval $[a, b]$ in $\mathbb{R}$. We will normal use $I = [0, 1]$. An endpoint of the path is $f(0)$ and $f(1)$.

Any two path $f, g : I \to \mathbb{R}^n$ is homotopic by a linear homotopy that is given by $F(t, s) = (1 - s)f(t) + sg(t)$ for $t, s \in [0, 1]^2$.

A *homotopy class* is an equivalence class of homotopic map relative to endpoints.

The *fundamental group* $\pi(X, x_0)$ at the *base point* x_0 is the set of homotopy class of path with both endpoints x_0.

The product in the fundamental group exists by joining. That is given two paths $f, g : I \to X$ with endpoints x_0, we define a path $g * f$ with endpoints x_0 by setting $g * f(t) = g(2t)$ if $t \in [0, 1/2]$ and $g * f(t) = f(2t - 1)$ if $t \in [1/2, 1]$. This induces a product $[f] * [g] = [f * g]$, which we need to verify to be well-defined with respect to the equivalence relation of homotopy. The constant path c_0 given by setting $c_0(t) = x_0$ for all t satisfies $[c_0] * [f] = [f] = [f] * [c_0]$. Denote it by 1_{x_0}. Given a path f, we can define an inverse path $f^{-1} : I \to X$ by setting $f^{-1}(t) = f(1 - t)$. We also obtain $[f^{-1}] * [f] = 1_{x_0} = [f] * [f^{-1}]$. By verifying $[f] * ([g] * [h]) = ([f] * [g]) * h$

for three paths with endpoints x_0, we see that the fundamental group is a group.

If we change the base to another point y_0 which is in the same path-component of X, we obtain an isomorphic fundamental group $\pi_1(X, y_0)$. Let γ be a path from x_0 to y_0. Then define $\gamma^* : [f] \in \pi_1(X, x_0) \mapsto [\gamma^{-1} * f * \gamma]$ which is an isomorphism. The inverse is given by $\gamma^{-1,*}$. This isomorphism does depend on γ and hence cannot produce a cannonical identification.

Theorem 2.1.3. *The fundamental group of a circle is isomorphic to $\mathbb{Z}$.*

This has a well-known corollary, the Brouwer fixed point theorem, that a self-map of a disk to itself always has a fixed point.

Given a map $f : X \to Y$ with $f(x_0) = y_0$, we define $f_* : \pi(X, x_0) \to \pi(Y, y_0)$ by $f_*([h]) = [f \circ h]$ for any path in X with endpoints x_0.

Theorem 2.1.4. *(Van Kampen Theorem) Given a space X covered by open subsets A_i such that any two or three of them meet at a path-connected set, $\pi(X, *)$ is a quotient group of the free product $*\pi(A_i, *)$. The kernel is the normal subgroup generated by $i_j^*(a) i_k^*(a)$ for any a in $\pi(A_i \cap A_j, *)$.*

For cell-complexes, these are useful for computing the fundamental group: If a space Y is obtained from X by attaching the boundary of 2-cells. Then $\pi(Y, *) = \pi(X, *)/N$ where N is the normal subgroup generated by "boundary curves" of the attaching maps.

A *bouquet of circles* is the quotient space of a union of n circles with one point from each identified with one another. Then the fundamental group at a basepoint x_0 is isomorphic to a free group of rank equal n. We will compute the fundamental group of surfaces later using this method.

2.1.3 *Covering spaces and discrete group actions*

Given a manifold M, a *covering map* $p : \tilde{M} \to M$ from another manifold $\tilde{M}$ is an onto map such that each point of M has a neighborhood O such that $p|p^{-1}(O) : p^{-1}(O) \to O$ is a homeomorphism for each component of $p^{-1}(O)$. Normally $\tilde{M}$ is assumed to be connected.

Consider $\mathbf{S}^1$ as the set of unit length complex numbers. The coverings of a circle $\mathbf{S}^1$ can be given by $f : \mathbf{S}^1 \to \mathbf{S}^1$ defined by sending $x \mapsto x^n$ These are finite to one covering maps. Define $\mathbb{R} \to \mathbf{S}^1$ by sending $t \to \exp(2\pi t i)$. Then this is an infinite covering.

Example 2.1.5. Consider a disk with interiors of disjoint smaller disks removed. Then draw mutually disjoint arcs from the boundary of the disk to the boundary of the smaller disk We remove mutually disjoint open regular neighborhoods of the disjoint arcs with boundary arcs again. Take the union of these arcs. Call these strips. Now we take the closures of each component of the complement. Let $D, I_1, I_2, ..., I_n$ denote the closures of the complement of the union of the strips and the strips. Let α_i^+, α_i^- the two boundary arcs of the trip I_i in the counter-clock wise direction. We take a product with a discrete countable set and label them by $D^i, I_1^i, ..., I_n^i$ for $i \in F$ for some countable set F. Then we select a permutation $k_j : F \to F$ for each $j = 1, 2, .., n$. We glue D^i with I_i along the arc α_i^+ and then we glue $D^{k_i(i)}$ along α_i^-. We do this for all arcs. Suppose we obtain a connected space. By sending $D^i \to D, I_i^j \to I_i$ by projections, we obtain a covering.

Another good example is the join of two circles: See [**(author?)** (Hatcher)] page 56–58

An important property of homotopy with respect to the covering space is the homotopy lifting property: Let $\tilde{M}$ be a covering of M. Given two homotopic maps f and g from a space X to M, if f lifts to $\tilde{M}$ and so does g. If we let $F : X \times I \to M$ be the homotopy, the map lifts to $F : X \times I \to \tilde{M}$. This is completely determined if the lift of f is specified.

For example, one can consider a path to be a homotopy for X a point. Any path in X lifts to a unique path in $\tilde{X}$ once the intial point is lifted.

Moreover, if two paths f, g are homotopic, and their initial point $\tilde{f}(0)$ and $\tilde{g}(0)$ of the lifts $\tilde{f}$ and $\tilde{g}$ are the same, then $\tilde{f}(1) = \tilde{g}(1)$. Using this idea, we can prove:

Theorem 2.1.6. *Given a map $f : Y \to M$ with $f(y_0) = x_0$, f lifts to $\tilde{f} : Y \to \tilde{M}$ so that $\tilde{f}(y_0) = \tilde{x}_0$ if $f_*(\pi(Y, y_0)) \subset p_*(\pi_*(\tilde{M}, \tilde{x}_0))$.*

An *isomorphism* of two covering spaces X_1 with a covering map $p_1 : X_1 \to X$ and X_2 with $p_2 : X_2 \to X$ is a homeomorphism $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. The *automorphism group* of a covering map $p : M' \to M$ is a group of homeomorphisms $f : M' \to M'$ so that $p \circ f = f$. We also use the term the *deck transformation group.* Each element is a *deck transformation* or a *covering automorphism.*

The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ on the right by path-liftings: For a point x_0 of M, we choose an inverse image $\tilde{x}_0$ in $\tilde{M}$. For a path γ in M with endpoint x_0, we define $\tilde{x}_0 \cdot \gamma = \tilde{\gamma}(1)$ for the lift $\tilde{\gamma}$ of γ with initial point $\tilde{\gamma}(0) = \tilde{x}_0$. This gives a right-action $\pi_1(M) \times \tilde{M} \to \tilde{M}$ since $\tilde{x} \cdot (\gamma * \delta) = (\tilde{x} \cdot \gamma) * \delta$.

A covering is *regular* if the covering map $p : M' \to M$ is a quotient map under the action of a discrete group Γ acting properly discontinuously and freely. Here M is homeomorphic to M'/Γ.

Given a covering map $p : \tilde{M} \to M$, there is a subgroup $p_*(\pi_1(\tilde{M}, \tilde{x}_0)) \subset \pi_1(M, x_0)$. Conversely, given a subgroup G of $\pi_1(M, x_0)$, we can construct a covering $\tilde{M}$ containing a point $\tilde{x}_0$ and a covering map $p : \tilde{M} \to M$ so that $p_*(\pi_1(\tilde{M}, \tilde{x}_0)) = G$.

One can classify covering spaces of M by the subgroups of $\pi(M, x_0)$. That is, two coverings M_1 with basepoint m_1 and the covering map p_1 and M_2 with basepoint m_2 and covering map p_2 of M with $p_1(m_1) = p_2(m_2) = x_0$ are isomorphic with a map sending m_1 to m_2 if the subgroups are the same. Thus, covering spaces can be ordered by subgroup inclusion relations. If the subgroup is normal, the corresponding covering is regular.

A manifold has a *universal covering*, i.e., a covering whose space has a trivial fundamental group. A universal cover covers every other coverings of a given manifold.

The universal covering $\tilde{M}$ of a manifold M has the covering automorphism group Γ isomorphic to $\pi_1(M)$. A manifold M equals $\tilde{M}/\Gamma$ for its universal cover $\tilde{M}$. Γ is a subgroup of the deck transformation group.

For example, let $\tilde{M}$ be $\mathbb{R}^2$ and T^2 be a torus. Then there is a map $p : \mathbb{R}^2 \to T^2$ sending (x, y) to $([x], [y])$ where $[x] = x \mod 2\pi$ and $[y] = y \mod 2\pi$.

Let M be a surface of genus 2. $\tilde{M}$ is homeomorphic to a disk. The deck transformation group can be realized as isometries of a hyperbolic plane. We will see this in more detail later.

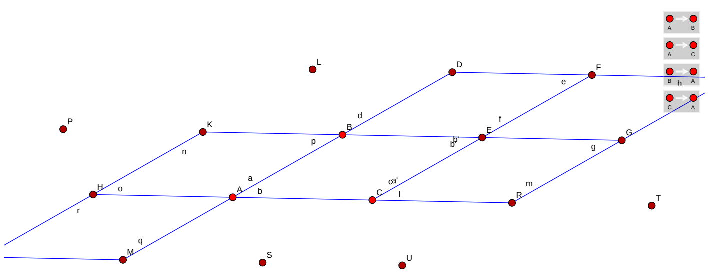


Fig. 2.1 A 2-torus as a quotient space of translation group of rank two

2.1.4 *Simplicial manifolds*

In this section, we will try to realize manifolds as a simplicial set.

An affine space A^n is a vector space $\mathbb{R}^n$ where we do not remember the origin. More, formally A^n equals $\mathbb{R}^n$ as a set but has an operation $\mathbb{R}^n \times A^n$ given by sending $(a,b) \to a+b$ for $a \in \mathbb{R}^n$ and $b \in A^n$ and satisfies $(a+(b+c)) = (a+b)+c$ for $a,b \in \mathbb{R}^n$ and $c \in A^n$. We can define the difference $b-a$ of two affine vectors a,b by setting $c \in \mathbb{R}^n$ be such that $c+a=b$.

If one take a point p as the origin, we can make A^n into a vector space $\mathbb{R}^n$ by a map $a \to a-p$ for all $a \in A^n$.

An $n+1$ points $v_1, ..., v_{n+1}$ in $\mathbb{R}^n$ is *affinely independent* if the set $v_i - v_1$ for $i = 2, ..., n+1$ is linearly independent as vectors. An n-simplex is a convex hull of an affinely independent $n+1$-points. An n-simplex is homeomorphic to B^n.

A *simplicial complex* is a locally finite collection S of simplices so that any face of a simplex is a simplex in S and the intersection of two elements of S is an element of S. The union is a topological set, which is said to be a *polyhedron*. We can define barycentric subdivisions by taking a varicentric subdivision for each simplex. A link of a simplex σ is the simplicial complex made up of simplicies opposite σ in a simplex containing σ.

An n-manifold X can be constructed by gluing n-simplices by face-identifications: Suppose X is an n-dimensional triangulated space. If the link of every p-simplex is homeomorphic to a sphere of $(n-p-1)$-dimension, then X is an n-manifold. If X is a simplicial n-manifold, we say X is orientable if we can give an orientations on each simplex so that over the common faces they extend each other.

2.1.4.1 *Surfaces*

We begin with a construction of a compact surface Given a polygon with even number of sides, we assign identification by labeling by alphabets $a_1, a_2, .., a_g, a_1^{-1}, a_2^{-1}, , , ,$ so that a_i means an edge labelled by a_i oriented counter-clockwise and a_i^{-1} means an edge labelled by a_i oriented clockwise. If a pair a_i and a_i or a_i^{-1} occur, then we identify them respecting the orientations.

Suppose we begin with a bigon.

- We divide the boundary into two edges and identify by labels a, a^{-1}. Then the result is a surface homeomorphic to a 2-sphere.

- We divide the boundary into two edges and identify by labels a, a. Then the result is homeomorphic to a projective plane.
- Suppose now we have a square, We identify the top segment with the bottom one and the right side with the left side. The result is homeomorphic to a 2-torus.

Any closed surface can be represented in this manner.

Let us be given a $4n$-gon. We label edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, ...a_n, b_n, a_n^{-1}, b_n^{-1}.$$

The result is a connected sum of n tori and is orientable. The genus of such a surface is n.

Suppose we are given a $2n$-gon. We label edges $a_1a_1a_2a_2....a_nb_n$. The result is a connected sum of n projective planes and is not orientable. The genus of such a surface is n.

The results are topological surfaces and a 2-dimensional simplicial manifold. We can remove the interiors of disjoint closed balls from the surfaces. The results are surfaces with boundary.

The fundamental group of a surface can now be computed using this identification. A surface is a cell complex starting from a 1-complex which is a bouquet of circles and attached with a cell.

$$\pi(S) = \{a_1, b_1, ..., a_g, b_g | [a_1, b_1][a_2, b_2]...[a_g, b_g]\}$$

for orientable S of genus g.

An Euler characteristic of a simplicial complex is given by $F - E + V$ where F denotes the number of 2-dimensional cells E the number of 1-dimensional cells, and V the number of 0-dimensional cells This is a topological invariant. We can count from the above identification picture that the Euler characteristic of an orientable compact surface of genus g with n boundary components is $2 - 2g - n$.

By a *simple curve* in a surface, we mean an imbedded interval. A *simple closed curve* in a surface is an imbedded circle. There play important roles in studying surfaces as Dehn and Nielson first discovered.

Let a 2-sphere be given a triangulation. A *pair of pants* is a complement of interior of three simplicial 2-cells in the sphere. It has three boundary components homeomorphic to circles. Moreover, a pair of pants can be obtained by identifying two hexagons in their alternating segments in pairs.

In fact, a closed orientable surface contains $3g-3$ disjoint simple closed curves so that the complement of its union is a disjoint union of open pairs of pants, i.e., spheres with three holes. Hence, the surface can be obtained as an identifying boundary components of the pairs of pants.

A pair of pants can have a simple closed curve imbedded in it but such a circle always bounds an annuli with a boundary component. Hence, a pair of pants can be built from a pair of pants and annuli by identification. One cannot build a pair of pants from a surface other than annuli and a single pair of pants. A pair of pants is an "elementary" surface in that any closed surface can be built from this piece by identifying boundary components.

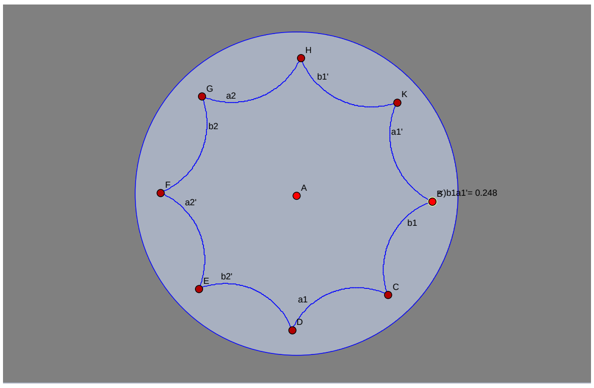


Fig. 2.2 A genus 2 surface as a quotient space of a disk

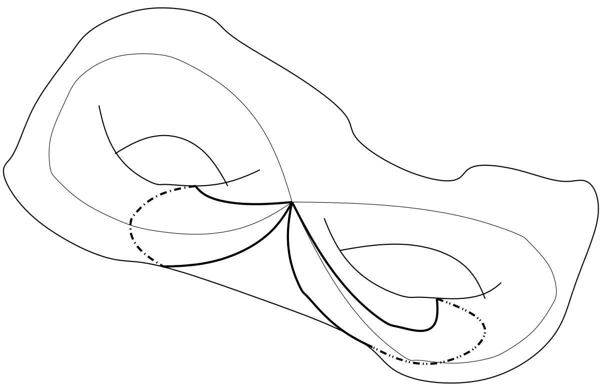


Fig. 2.3 A genus 2-surface patched up

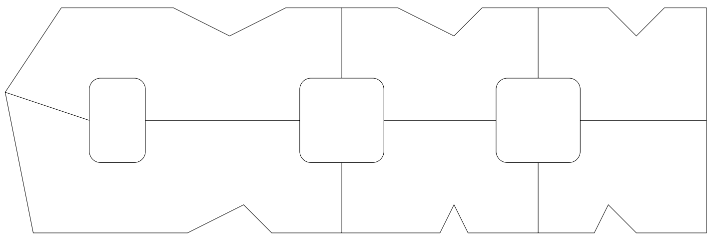


Fig. 2.4 A genus n surface as a double of unions of hexagons

2.2 Lie groups

2.2.1 *Lie groups*

A Lie group can be thought of as a space of symmetries of some space. More formally, a Lie group is a manifold with a group operation $\circ : G \times G \to G$ that satisfies

- $\circ$ is smooth.
- the inverse $\iota : G \to G$ is smooth also.

From $\circ$, we form a homomorphism $G \to Diff(G)$ given by $g \mapsto L_g$ and $L_g : G \to G$ is a diffeomorphism given by a left-multiplication $L_g(h) = gh$. Since we have $L_(gh) = L_g \circ L_h$, this is a homomorphism.

As example, we have:

- The permutation group of a finite set form a 0-dimensional manifold, which is a finite set, and a countable infinite group with discrete topology.
- $\mathbb{R}, \mathbb{C}$ with $+$ as an operation. ($\mathbb{R}^+$ with $+$ is merely a Lie semi-group.)
- $\mathbb{R} - \{O\}, \mathbb{C} - \{O\}$ with $*$ as an operation.
- $T^n = \mathbb{R}^n/\Gamma$ with $+$ as an operation and O as the equivalence class of $(0, 0, ..., 0)$ and Γ is a group of translations by integeral vectors. (The three are abelian ones.)

We go to the noncommutative groups.

- The general linear group is given by $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$: Here, $GL(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. The multiplication is smooth since the coordinate product has a polynomial expressions.

- The special linear group is given as $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R})|det(A) = 1\}$: The restriction by a polynomial gives us smooth submanifold of $GL(n,\mathbb{R})$. The multiplication are also the restrictions.
- The orthogonal group is given by $O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R})|A^T A = \mathrm{I}\}$. This is another submanifold formed by polynomials.
- The Euclidean isometry group is given by $Isom(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n|T(x) = Ax + b \text{ for } A \in O(n,\mathbb{R}), b \in \mathbb{R}^n\}$. This is a semiproduct group.

Let us state some needed facts.

- A products of Lie groups form a Lie group.
- A covering space of a connected Lie group form a Lie group.
- A *Lie subgroup* of a Lie group is a subgroup that is a Lie group with the induced operation and is a submanifold. For example , consider
  - $O(n) \subset SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$.
  - $O(n) \subset Isom(\mathbb{R}^n)$.

A *homomorphism* $f : G \to H$ of two Lie groups G and H is a smooth map that is a group homomorphism. The above inclusion maps are homomorphisms. The kernel of a homomorphism is a closed normal subgroup. Hence it is a Lie subgroup also. If G and H are simply connected, f induces a unique homomorphism of Lie algebra of G to that of H which equals Df and conversely. (See Subsection 2.2.2 for the definition of the Lie algebras and their homomorphsms.)

2.2.2 *Lie algebras*

A Lie algebra is a real or complex vector space V with an operation $[,] : V \times V \to V$ that satisfies:

- $[x, x] = 0$ for every $x \in V$. (Thus, $[x, y] = -[y, x]$.)
- the Jacobi identity $[x, [y, z]]+[z, [x, y]]+[y, [z, x]] = 0$ for all $x, y, z \in V$.

Examples:

- Sending $V \times V$ to the zero-element O is a Lie algebra. This is defined to be the *abelian Lie algebras.*

- The direct sums of Lie algebras is a Lie algebra.
- A subalgebra is a subspace closed under $[,]$.
- An ideal K of V is a subalgebra such that $[x, y] \in K$ for $x \in K$ and $y \in V$.

A homomorphism of a Lie algebra is a linear map preserving $[,]$. The kernel of a homomorphism is an ideal.

2.2.3 *Lie groups and Lie algebras*

Let G be a Lie group. For an element $g \in G$, a *left translation* $L_g : G \to G$ is given by $x \mapsto g(x)$. A *left-invariant vector field* of G is a vector field so that the left translation leaves it invariant, i.e., $dL_g(X(h)) = X(gh)$ for $g, h \in G$.

- The set of left-invariant vector fields form a vector space under addition and scalar multiplication and is vector-space isomorphic to the tangent space at I. Moreover, $[,]$ is defined as vector-fields brackets. This forms a Lie algebra.
- The Lie algebra of G is the the Lie algebra of the left-invariant vector fields on G.

A Lie algebra of an abelian Lie group is abelian.

Let $gl(n, \mathbb{R})$ denote the $M_n(\mathbb{R})$ with $[,] : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $[A, B] = AB - BA$ for $A, B \in M_n(\mathbb{R})$. The Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to $gl(n, \mathbb{R})$:

- For X in the Lie algebra of $GL(n, \mathbb{R})$, we can define a flow generated by X and a path $X(t)$ along it where $X(0) = \mathrm{I}$.
- We obtain an element of $gl(n, \mathbb{R})$ by taking the derivative of $X(t)$ at 0 seen as a matrix.
- Now, we show that the brackets are preserved. That is, a vector-field bracket becomes a matrix bracket by the above map.

Given X in the Lie algebra $\mathfrak{g}$ of G, there is an integral curve $X(t)$ through I. We define the *exponential map* $\exp : \mathfrak{g} \to G$ by sending X to $X(1)$. The exponential map is defined everywhere, smooth, and is a diffeomorphism near O. With some work, we can show that the matrix exponential defined by

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{1}{k!} A^k$$

is the exponential map exp : $gl(n, \mathbb{R}) \to GL(n, \mathbb{R})$. (See in for example [**?**
)])

2.2.4 *Lie group actions*

A *left* Lie group G*-action* on a smooth manifold X is given by a smooth map $k : G \times X \to X$ so that $k(e, x) \mapsto x$ and $k(gh, x) = k(g, k(h, x))$. Normally, $k(g, x)$ is simply written $g(x)$. In other words, denoting by $Diffeo(X)$ the group of diffeomorphisms of X, k gives us a homomorphism $k'\ G \to Diffeo(X)$ so that $k'(gh)(x) = k'(g)(k'(h)h(x))$ and $k'(I) = I_X$. This is said to be the *left-action*. (We will not use notations k and k' in most cases.)

- A *right action* satisfies $(x)(gh) = ((x)g)h$ or more precisely, $(gh)(x) = (h(g(x))$.
- Each Lie algebra element correspond to a vector field on X by using a vector field.
- The action is *faithful* if $g(x) = x$ for all x, then g is the identity element of G. This means that only g that correspond to the identity on X is e.
- The action is *transitive* if given two points $x, y \in X$, there is $g \in G$ such that $g(x) = y$.

As examples, consider

- $GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$.
- $PGL(n+1, \mathbb{R})$ acting on $\mathbb{R}P^n$.

2.3 Pseudo-group and $\mathcal{G}$-structures

In this section, we introduce pseudo-groups. Topological manifolds and its submanifolds are very wild and complicated objects to study as the topologist in 1950s and 1960s found out. The pseudo-groups will be used to put "calming" structures on manifolds.

Often the structures will be modelled on some geometries. We are mainly interested in classical geometries, Clifford-Klein geometries. We will be concerned with Lie group G acting on a manifold M. Most obvious ones are euclidean geometry where G is the group of rigid motions acting on the euclidean space $\mathbb{R}^n$. The spherical geometry is one where G is the group $O(n+1)$ of orthogonal transformations acting on the unit sphere $\mathbf{S}^n$.

Topological manifolds form too large category to handle. To restrict the local property more, we introduce *pseudo-groups*. A *pseudo-group* $\mathcal{G}$ on a topological space X is the set of homeomorphisms between open sets of X so that

- The domains of $g \in \mathcal{G}$ cover X.
- The restriction of $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.
- The composition of two elements of $\mathcal{G}$ when defined is in $\mathcal{G}$.
- The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
- If $g : U \to V$ is a homeomorphism for U, V open subsets of X. If U is a union of open sets U_α for $\alpha \in I$ for some index set I such that $g|U_\alpha$ is in $\mathcal{G}$ for each α, then g is in $\mathcal{G}$.

Let us give some examples:

- The trivial pseudo-group is one where every element is a restriction of the identity $X \to X$.
- Any pseudo-group contains a trivial pseudo-group.
- The maximal pseudo-group of $\mathbb{R}^n$ is TOP, the set of all homeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group C^r, $r \geq 1$, of the set of C^r-diffeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group PL of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.
- A (G, X)-*pseudo group* is defined as follows. Let G be a Lie group acting on a manifold X. Then we define the pseudo-group as the set of all pairs $(g|U, U)$ for $g \in G$ where U is the set of all open subsets of X. (See Subsection **??** and Chapter 3 for details on Lie groups and their actions.)
- The group isom$(\mathbb{R}^n)$ of rigid motions acting on $\mathbb{R}^n$ or orthogonal group $O(n+1, \mathbb{R})$ acting on $\mathbf{S}^n$ give examples.

2.3.1 $\mathcal{G}$-*manifold*

A $\mathcal{G}$-manifold is obtained as a manifold glued with special type of gluings only in $\mathcal{G}$: Let $\mathcal{G}$ be a pseudo-group on $\mathbb{R}^n$. A $\mathcal{G}$-*manifold* is a n-manifold M with a maximal $\mathcal{G}$-atlas.

A $\mathcal{G}$-*atlas* is a collection of charts (imbeddings) $\phi : U \to \mathbb{R}^n$ where U is an open subset of M such that whose domains cover M and any two charts are $\mathcal{G}$-compatible.

- Two charts $(U, \phi), (V, \psi)$ are *$\mathcal{G}$-compatible* if the transition map $$\gamma = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \in \mathcal{G}.$$

A set of $\mathcal{G}$-atlases is a partially ordered set under the ordering relation given by inclusion relation. Two $\mathcal{G}$-atlases are compatible if any two charts in the atlases are $\mathcal{G}$-compatible. In this case, the union is another $\mathcal{G}$-atlas. One can choose a locally finite $\mathcal{G}$-atlas from a given maximal one and conversely. By Zorn's lemma, the set of compatible $\mathcal{G}$-atlases has a unique maximal $\mathcal{G}$-atlas.

Under compatibility relation, the set of all $\mathcal{G}$-structures is partitioned into equivalence classes. We define the *$\mathcal{G}$-structure* on M as a maximal $\mathcal{G}$-atlas or as an equivalence class in the above partition.

The manifold X is trivially a $\mathcal{G}$-manifold if $\mathcal{G}$ is a pseudo-group on X. A topological manifold has TOP-structure. A C^r-manifold is a manifold with a C^r-structure. A *differentiable manifold* is a manifold with C^∞-structure. A *PL-manifold* is a manifold with a PL-structure.

A $\mathcal{G}$-map $f : M \to N$ for two $\mathcal{G}$-manifolds is a local homeomorphism so that if f sends a domain of a chart ϕ into a domain of a chart ψ, then
$$\psi \circ f \circ \phi^{-1} \in \mathcal{G}.$$
That is, f is an element of $\mathcal{G}$ locally up to charts.

Given two manifolds M and N, let $f : M \to N$ be a local homeomorphism. If N has a $\mathcal{G}$-structure, then so does M so that the map is a $\mathcal{G}$-map. A $\mathcal{G}$-atlas is given on M by taking open sets so that it maps into charts in N under f and then use the induced chart. The $\mathcal{G}$-structure is said to be the *induced $\mathcal{G}$-structure*. (A class of examples such as θ-annuli and π-annuli that arises in the study of complex projective and real projective surfaces.)

Let Γ be a discrete group of $\mathcal{G}$-homeomorphisms of M acting properly and freely. Then M/Γ has a $\mathcal{G}$-structure. The charts will be the charts of the lifted open sets. The $\mathcal{G}$-structure here is said to be the *quotient $\mathcal{G}$-structure.*

The torus T^n has a C^r-structure and a PL-structure since so does $\mathbb{R}^n$ and the each element of the group of translations all preserve these structures.

Given a pair (G, X) of Lie group G acting on a manifold X, a *(G, X)-structure* is a $\mathcal{G}$-structure and a *(G, X)-manifold* is a $\mathcal{G}$-manifold where $\mathcal{G}$ is the (G, X)-pseudo group.

A *euclidean manifold* is a $(\mathrm{isom}(\mathbb{R}^n), \mathbb{R}^n)$-manifold.

A *spherical manifold* is a $(O(n+1), \mathbf{S}^n)$-manifold.

2.4 Differential geometry and $\mathcal{G}$-structures

We wish to understand geometric structures in terms of differential geometric setting; i.e., using bundles, connections, and so on. since such an understanding help us to see the issues in different ways. Actually, this is not central to the lectures. However, we should try to relate to the traditional fields where our subject can be studied in another way.

2.4.1 *Riemannian manifolds*

A differentiable manifold has a Riemannian metric, i.e., inner-product at each tangent space smooth with respect coordinate charts. Such a manifold is said to be a Riemannian manifold.

An isometric immersion (imbedding) of a Riemannian manifold to another one is a (one-to-one) map that preserve the Riemannian metric. We will be concerned with isometric imbedding of M into itself usually. A length of an arc is the value of an integral of the norm of tangent vectors to the arc. This gives us a metric on a manifold. An isometric imbedding of M into itself is an isometry always. A geodesic is an arc minimizing length locally.

A sectional curvature of a Riemannian metric along a 2-plane is given as the rate of area growth of disks (An exact formula exists.) To be filled in!!!

A constant curvature manifold is one where the sectional curvature is identical to a constant in every planar direction at every point.

- A euclidean space E^n with the standard norm metric has a constant curvature $= 0$.
- A sphere $\mathbf{S}^n$ with the standard induced metric from $\mathbb{R}^{n+1}$ has a constant curature $= 1$.
- Given a discrete torsion-free subgroup Γ of the isometry group of E^n (resp. $\mathbf{S}^n$). Then E^n/Γ (resp. $\mathbf{S}^n/\Gamma$) is a manifold with a constant curvature $= 0$ (resp. 1).

2.4.2 *Principal bundles and connections, flat connections*

Let M be a manifold and G a Lie group. A *principal fiber bundle* P over M with a group G is the object satisfying

- P is a manifold.
- G acts freely on P on the right given by a smooth map $P \times G \to P$.

- $M = P/G$ and the map $\pi : P \to M$ is differentiable.
- P is locally trivial. That is, there is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ for at least one neighborhood U of any point of M.

We say that P the *bundle space*, M the *base space*. $\pi^{-1}(x)$ a fiber which also equals $\pi^{-1}(x) = \{ug|g \in G\}$ for any choice of $u \in \pi^{-1}(x)$. G is said to be the *structure group*.

As an example, consider: $L(M)$ the set of all frames of the tangent bundle $T(M)$. One can give a topology on $L(M)$ so that sending a frame to its base point the smooth quotient map $L(M) \to M$. $GL(n, \mathbb{R})$ acts freely on $L(M)$. We can verify that $\pi : L(M) \to M$ is a principal bundle.

Given a collection of open subsets U_α covering M, a bundle can be constructed by a collection of mappings

$$\{\phi_{\beta,\alpha} : U_\alpha \cap U_\beta \to G | U_\alpha, U_\beta\}$$

satisfying

$$\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}$$

for any triple $U_\alpha, U_\beta, U_\gamma$. Then form $U_\alpha \times G$. For any pair $U_\alpha \times G$ and $U_\beta \times G$, identify by $\tilde{\phi}_{\beta,\alpha} : U_\alpha \times G \to U_\beta \times G$ given by $(x, g) \mapsto (x, \phi_{\beta,\alpha}(g))$. The quotient space is a principal bundle over M.

A principal bundle over M with the structure group G is often denoted by $P(G, M)$. Given two Lie groups G and G', and a monomorphism $f : G' \to G$, a map $f : P(G', M) \to P(G, M)$ inducing identity $M \to M$ is called a *reduction* of the structure group G to G'. There maybe many reductions for given G' and G. We say that $P(G, M)$ is *reducible to* $P(G', M)$ if and only if $\phi_{\alpha,\beta}$ can be taken to be in G'. (See [**(author?)** (Kobayashi and Nomizu)] and [**(author?)** (Bishop)] for details.)

2.4.2.1 *Associated bundles*

Let F be a manifold with a left-action of G. G acts on $P \times F$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in P, x \in F.$$

Form the quotient space $E = P \times_G F$. with a map π_E is induced and we can verify that $\pi_E^{-1}(U)$ is identifiable with $U \times F$ up to making some choices of sections U to P. The space E is said to be the *associated bundle* over M with M as the *base space*. The *structure group* is the same G. Again there is a induced quotient map $\pi : E \to M$ with fiber $\pi^{-1}(x)$ diffeomorphic to F for any $x \in M$.

Here E can also be built from a cover U_α of M by taking $U_\alpha \times F$ and pasting by appropriate diffeomorphisms of F induced by elements of g as above.

The tangent bundle $T(M)$ is an example. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ on the left. Let $F = \mathbb{R}^n$. We obtain $T(M)$ as $L(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$. A tensor bundle $T^r_s(M)$ is another example. $GL(n, \mathbb{R})$ acts on $T^r_s(\mathbb{R}^n)$. Let $F = T^r_s(\mathbb{R})$. Then we obtain $T^r_s(M)$ as $L(M) \times_{GL(n,\mathbb{R})} T^r(\mathbb{R}^n)$.

2.4.2.2 *Connections*

Let $P(M, G)$ be a principal bundle. A *connection* is a decomposion of each $T_u(P)$ for each $u \in P$

- $T_u(P) = G_u \oplus Q_u$ where G_u is a subspace tangent to the fiber. (G_u is said to be the *vertical space* and Q_u the *horizontal space*.)
- $Q_{ug} = (R_g)_* Q_u$ for each $g \in G$ and $u \in P$.
- Q_u depend smoothly on u.

Let $\mathfrak{g}$ denote the Lie algebra of G. More formally, we define a connection as a $\mathfrak{g}$-valued form ω on P is given as $T_u(P) \to G_u$ given by taking the vertical component of each tangent vector of P: We can define a *connection* as a smooth $\mathfrak{g}$-valued form ω.

- $\omega(A^*) = A$ for every $A \in \mathfrak{g}$ and A^* the fundamental vector field on P generated by A, i.e., the vector field tangent to the one parameter group of diffeomorphisms on P generated by the action of $\exp(tA) \in G$ at $t = 0$.
- $(R_g)^*\omega = ad(g^{-1})\omega$.

A *horizontal* lift of a piecewise-smooth path τ on M is a piecewise-smooth path τ' lifting τ so that the tangent vectors are all horizontal. A horizontal lift is determined once the initial point is given.

- Given a curve on M with two endpoints, the lifts defines a parallel displacement between fibers above the two endpoints. (commuting with the right G-actions).
- Fixing a point x_0 on M, this defines a holonomy group.
- The curvature of a connection is a measure of how much a horizontal lift of small loop in M differ from a loop in P.
- For the flat connections, we can lift homotopically trivial loops in M^n to loops in P. Thus, the flatness is equivalent to local lifting

of a small coordinate chart of M to horizontal sections in P.
- A flat connection on P gives us a smooth foliation of dimension n transversal to the fibers.

The associated bundle E also inherits a *connection*, i.e., a splitting of the tangent space E into vertical space and horizontal space. Here again, the vertical space is obtained as G-orbits. Again given a curve on M, horizontal lifings to E make sense and parallel displacements between fibers. The flatness is also equivalent to the local lifting property, and The flat connection on E gives us a smooth foliation of dimension n transversal to the fibers.

Check G acts on E on the left? Connection now left invariant? Some checking to do.. Jul 17

An *affine frame* in a vector (or affine) space A^n is a set of $n+1$ points $a_0, a_1, ..., a_n$ so that $a_1 - a_0, a_2 - a_0, ..., a_n - a_0$ form a linear frame. This assignments gives us the canonical map from the space of affine frames $A(A^n)$ to linear frames $L(A^n)$. An affine group $A(n, \mathbb{R})$ acts on $A(A^n)$ also by sending $(a_0, a_1, .., a_n)$ to $(L(a_0), L(a_1), ..., L(a_n))$ for an affine automorphism $L : A^n \to A^n$.

An *affine connection* on a manifold M is defined as follows. An affine frame over M is an affine frame on a tangent space of a point of M, treating as an affine space. The set of all affine frames over a manifold form a manifold of higher dimension. Let $A(M)$ be the space of affine frames over M with the affine group $A(n, \mathbb{R})$ acting on it fiberwise on the left.

- The Lie algebra $a(n, \mathbb{R})$ is a semi-direct sum of $M(n, \mathbb{R})$ and $\mathbb{R}^n$.
- There is a natural map $A(M) \to L(M)$ where $L(M)$ is the set of linear frames over M which is given by the natural map $A(E^n) \to L(E^n)$.
- An affine connection on M is a linear connection plus the canonical $\mathbb{R}^n$-valued 1-form. The curvature of the affine connection is the sum of the curvature of the linear connection and the torsion.

A nice example is when M is a 1-manifold, say an open interval I. Then P is $I \times G$ and the associated bundle is $I \times X$. A connection is simply given as an infinitesimal way to connect each fiber by a left multiplication by an element of G. In this case, a connection is flat always and $I \times G$ and $I \times X$ are fibered by open intervals transversal to the fibers.

If M is a circle, this gives P becomes a mapping circle with fiber G and E a mapping circle with fiber E. Now, such space can be classified by a map $\pi_1(\mathbf{S}^1) \to G$.

For the affine connection, for $M = I$, we use $G = A(1, \mathbb{R})$ and $X = \mathbb{R}$. Then E is now an annulus $I \times \mathbb{R}$. An affine connection gives a foliation on annulus transversal to $\mathbb{R}$ and is invariant under translation in $\mathbb{R}$-direction.

Even for higher-dimensional manifolds, we can think of connection as 1-dimensional ones over each paths. The local dependence on paths is measured by the curvature.

Summary: A connection gives us a way to identify fibers given paths on X-bundles over M. The flatness gives us locally consistent identifications.

2.4.2.3 *The principal bundles and G-structures.*

I need to elaborate on this more..

Given a manifold M of dimension n, a Lie group G acting on a manifold X of dimension n. We form a principal bundle P over a manifold M and then the associated bundle E fibered by X with a flat connection. Suppose we can choose a section $f : M \to E$ which is transverse everywhere to the foliation given by the flat connection. This gives us a (G, X)-structure. The main reason is that the section f sends an open set of M to a transversal submanifold to the foliation. Locally, the foliation gives us a projection to X. The composition gives us charts. The charts can are compatible since E has left-action.

Coversely a (G, X)-structure gives us P, E, f and the flat connection.

We will elaborate this later when we are studying orbifolds and geometric structures.

2.5 Notes to Chapter 1

Course home page: `math.kaist.ac.kr/~schoi/dgorb.htm` and `http://www.is.titech.ac.jp/~schoi/dgorb.htm`

Chapter 0 and 1 of [**(author?)** (Hatcher)] and [**(author?)** (Munkres)] and [**(author?)** (Warner)] are good source of preliminary knowledges here. [**(author?)** (Do Carmo)], [**(author?)** (Kobayashi and Nomizu)], and [**(author?)** (Bishop)] give us good knowledge of connections, curvature, and Riemannian geometry.

Chapter 3

Lie groups and geometry

3.1 Introduction

In this section, we will introduce basic materials in Lie group theory and geometry and discrete group actions on the geometric spaces.

Geometry will be introduced as in Klein's Erlangen program. Hyperbolic geometry will be given empasis by detailed descriptions of models. Finally, we discuss the discrete group actions, Poincare polyhedron theorem and the crystallographic group theory.

- Euclidean geometry
- Spherical geometry
- Affine geometry
- Projective geometry
- Conformal geometry: Poincare extensions
- Hyperbolic geometry
  - Lorentz group
  - Geometry of hyperbolic space
  - Beltrami-Klein model
  - Conformal ball model
  - The upper-half space model
- Discrete groups.

3.2 Geometries

We will now describe classical geometries from Lie group action perspectives.

3.2.1 *Euclidean geometry*

The Euclidean space is $\mathbb{R}^n$ and the group $Isom(\mathbb{R}^n)$ of rigid motions is generated by $O(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.

A system of linear equations gives us a subspace (affine or linear). Hence, we have a notion of points, lines, planes, and angles. Notice that these notions are invariantly defined under the group of rigid motions. These give us the set theoretical model for the Euclidean axioms.

3.2.2 *Spherical geometry*

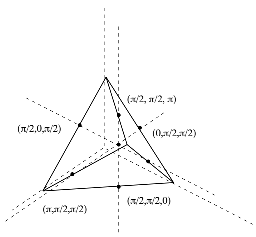
Let us consider the unit sphere $\mathbf{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$. The transformation group is $O(n+1)$.

Many great sphere exists and they are subspaces as they are given by homogeneous system of linear equations in $\mathbb{R}^{n+1}$. The lines are replaced by great circles and lengths and angles are also replaced.

Many spherical triangle theorems exist... `http://mathworld.wolfram.com/SphericalTrigonometry.html` Such a triangle is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$\theta_0 + \theta_1 + \theta_2 > \pi \tag{3.1}$$

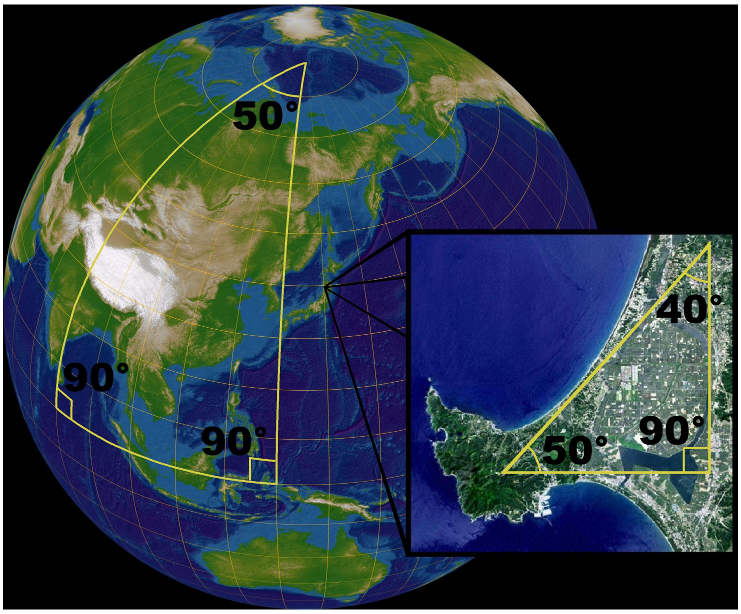
$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \tag{3.2}$$



New diagrams for two.

3.2.3 *Affine geometry*

A vector space $\mathbb{R}^n$ becomes an affine space by forgetting about the previliges of the origin. An affine transformation of $\mathbb{R}^n$ is one given by $x \mapsto Ax + b$ for $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. This notion is more general than that of rigid motions.



The Euclidean space $\mathbb{R}^n$ with the group $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$ of affine transformations form the affine geometry. Of course, angles and lengths do not make sense. But the notion of lines exists. Also, affine subspaces that is a linear space translated by a vector make sense.

The set of three points in a line has an invariant based on ratios of lengths.

3.2.4 *Projective geometry*

Projective geometry was first considered from fine art. Desargues (and Kepler) first considered points at infinity from mathematical point of view. Poncelet first added infinite points to the euclidean plane.

Here, the transformations are generated by perspectives, i.e., transformation of projecting one plane to another plane by light ray from a point source. Projective transformations are compositions of perspectivities. Often, they send finite points to infinite points and vice versa. (i.e., two planes that are not parallel). Therefore, we need to add infinite points while the added points are same as ordinary points up to projective transformations.

Lines have well defined infinite points and are really circles topologically because we added infinite point at each direction. Some notions such as angles and lengths lose meanings. However, many interesting theorems can be proved. Also, theorems always come in dual pairs by switching lines to points and vice versa. Duality of theorems plays an interesting role. (See for an interactive course: `http://www.math.poly.edu/courses/projective_geometry/` and `http://demonstrations.wolfram.com/TheoremeDePappusFrench/`, `http://demonstrations.wolfram.com/TheoremeDePascalFrench/`, `http://www.math.umd.edu/~wphooper/pappus9/pappus.html`, `http://www.math.umd.edu/~wphooper/pappus/`)

A formal definition with topology is given by Felix Klein using homogeneous coordinates. The projective space $\mathbb{R}P^n$ is $\mathbb{R}^{n+1} - \{O\}/\sim$ where $\sim$ is given by $v \sim w$ if $v = sw$ for $s \in \mathbb{R}$. Each point is given a homogeneous coordinates: $[v] = [x_0, x_1, ..., x_n]$ where two homogeneous coordinates are equal if they differ only by a nonzero scalar. That is $[x_0, x_1, ..., x_n] = [\lambda x_0, \lambda x_1, ..., \lambda x_n]$ for $\lambda \in \mathbb{R} - \{0\}$. The projective transformation group $\mathrm{PGL}(n+1, \mathbb{R})$ is defined as $GL(n+1, \mathbb{R})/\sim$ where $g \sim h$ for $g, h \in GL(n+1, \mathbb{R})$ if $g = ch$ for a nonzero constant c. We can also see that the group eqauls the quotient group $SL_{\pm}(n+1, \mathbb{R})/\{\mathrm{I}, -\mathrm{I}\}$. of the group $SL_{\pm}(n+1, \mathbb{R})$ of determinant ± 1. Now $\mathrm{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{R}P^n$ by each element sending each ray to a ray using the corresponding general linear maps. Each element of g of $\mathrm{PGL}(n+1, \mathbb{R})$ acts by $[v] \mapsto [g'(v)]$ for a representative g' in $GL(n+1, \mathbb{R})$ of g.

That is, given a basis B of $n+1$ vectors $v_0, v_1, ..., v_n$ of $\mathbb{R}^{n+1}$ for a point v, we let $[v]_B = [\lambda_0, \lambda_1, ..., \lambda_n]_B$ if we can write $v = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n$. Here, $[\lambda_0, ..., \lambda_n]_B = [c\lambda_0, c\lambda_1, ..., c\lambda x_n]_B$ for $c \in \mathbb{R} - \{0\}$.

Also any homogeneous coordinate change can be viewed as induced by a linear map: That is, $[v]_B$ has same homogeneous coordinates as $[Mv]$ where M is the coordinate change linear map so that $Mv_i = e_i$ for $i = 0, 1, .., n$.

- For us $n = 2$ is important. Here we have a familiar projective plane as topological type of $\mathbb{R}P^2$, which is a Mobius band with a disk filled in at the boundary. See `http://www.geom.uiuc.edu/zoo/toptype/pplane/cap/`.
- The affine geometry can be "imbedded": $\mathbb{R}^n$ can be identified with the set of points in $\mathbb{R}P^n$ where x_0 is not zero, i.e., the set of points $\{[1, x_1, x_2, ..., x_n]\}$. This is called an affine patch. The subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ fixing $\mathbb{R}^n$ is precisely $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$ as can be seen by computations.
- The subspace of points $\{[0, x_1, x_2, ..., x_n]\}$ is the complement home-

omorphic to $\mathbb{R}P^{n-1}$. This is the set of infinities, i.e., directions in $\mathbb{R}P^n$.

- From affine geometry, one can construct a unique projective geometry and conversely using this idea. (See Berger [**?**)] for the complete abstract approach.)
- The independence of points are defined. The dimension of a subspace is the maximal number of independent set minus 1.
- A subspace is the set of points whose representative vectors satisfy a homogeneous system of linear equations. The subspace in $\mathbb{R}^{n+1}$ corresponding to a projective subspace in $\mathbb{R}P^n$ in a one-to-one manner while the dimension drops by 1.
- A hyperspace is given by a single linear equation. The complement of a hyperspace can be identified with an affine space.
- A line is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. Acutally a line is $\mathbb{R}P^1$ or a line $\mathbb{R}^1$ with a unique infinity. A point on a line is given a homogeneous coordinates $[s, t]$ where $[s, t] \sim [\lambda s, \lambda t]$ for $\lambda \in \mathbb{R} - \{O\}$.

The projective geometry has well-known invariant called cross ratios eventhough lengths of immersed geodesics and angles between smooth arcs are not invariants. (However, we do note that the properties of angles or length $< \pi, = \pi$, or $> \pi$ are invariant properties.)

- The cross ratio of four points x, y, z, and t on a line is defined as follows. There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. Thus $b = b(x, y, z, t)$ is the cross-ratio. Thus, it is necessary that at least three points x, y, z are mutually distinct.
- If the four points are on $\mathbb{R}^1$, the cross ratio is given as
$$(x, y; z, t) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$
if we write
$$x = [1, z_1], y = [1, z_2], z = [1, z_3], t = [1, z_4]$$
by a some coordinate change.
- One can define cross ratios of four hyperplanes meeting in a projective subspace of codimension 2. By duality, they correspond to four points on a line.

3.2.4.1 *Oriented projective geometry*

Note that $\mathbf{S}^n$ double covers $\mathbb{R}P^n$. Moreover, the group $\mathrm{SL}_\pm(n+1,\mathbb{R})$, i.e., linear maps of $\mathbb{R}^{n+1}$ with determinant ± 1, maps to $\mathrm{PGL}(n+1,\mathbb{R})$ with discrete kernels in the center. Then $(\mathbf{S}^n, \mathrm{SL}_\pm(n+1,\underline{)})$ defines a geometry said to be *oriented projective geometry.*

This is an old idea actually, and there are number of advantages working in this spaces.

Each point is given a homogeneous coordinates: $[v] = [x_0, x_1, ..., x_n]$ where two homogeneous coordinates are equal if they differ only by a nonzero scalar, i.e., $[x_0, x_1, ..., x_n] = [\lambda x_0, \lambda x_1, ..., \lambda x_n]$ for $\lambda \in \mathbb{R}, \lambda > 0$.

Two points are *antipodal* if their homogeneous coordinates are minus of each other.

Subspaces are defined by linear equations as above. A great circle is a subspace of dimension 1. A set of a point is not a subspace. A pair of antipodal points is a subspace. Independence is defined as above.

Again a great circle has a homogeneous coordinate: A great circle is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. A point on a great circle is given a homogeneous coordinates $[s, t]$ where $[s, t] \sim [\lambda s, \lambda t]$ for $\lambda \in \mathbb{R}, \lambda > 0$. Cross ratio can be defined on four distinct points (x, y, z, t) on a great circle with say the first homogeneous coordinates positive. There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. Then $b = b(x, y, z, t)$ is the cross-ratio.

3.2.5 *Conformal geometry*

We can introduce two symmetries of $\mathbb{R}^n$. The first class is the set of reflections of $\mathbb{R}^n$. Let the hyperplane $P(a, t)$ given by $a \cdot x = b$. Then the reflection about $P(a, t)$ is given by $\rho(x) = x + 2(t - a \cdot x)a$. The second class is the set of inversions. Let the hypersphere $S(a, r)$ be given by $|x - a| = r$. Then the inversion about $S(a, r)$ is given by $\sigma(x) = a + (\frac{r}{|x-a|})^2(x - a)$.

We compactify $\mathbb{R}^n$ to $\hat{\mathbb{R}}^n = \mathbf{S}^n$ by adding infinity. This can be accomplished by a stereographic projection from the unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ from the northpole $(0, 0, ..., 1)$. Taking the inverse image of $\mathbb{R}^n$ in $\mathbf{S}^n$, we obtain a copy of $\mathbb{R}^n$ in $\mathbf{S}^n$. The usual differentiable structure of $\mathbf{S}^n$ extends that of imbedded $\mathbb{R}^n$. Since the stereographic map preserves angles, the angles of $\mathbb{R}^n$ agree with those of the copy in $\mathbf{S}^n$ with the standard metric. The reflections and inversions of $\mathbb{R}^n$ become diffeomorphisms of the copy in $\mathbf{S}^n$, which extend to unique real analytic diffeomorphisms of $\mathbf{S}^n$ respectively, that is, their Jacobians are nowhere zero. Since the maps preserve angles almost everywhere, they are do so everywhere by a limiting argument. Thus, these reflections and inversions induce conformal homeomorphisms of $\hat{\mathbb{R}}^n = \mathbf{S}^n$; that is, they preserve angles.

- The group of transformations generated by these homeomorphisms is called the *Mobius transformation group*.
- They form the conformal transformation group of $\hat{\mathbb{R}}^n = \mathbf{S}^n$.
- For $n = 2$, $\hat{\mathbb{R}}^2$ is the Riemann sphere $\hat{\mathbb{C}}$ and a Mobius transformation is a either a fractional linear transformation of form

$$z \mapsto \frac{az+b}{cz+d}, ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

  or a fractional linear transformation pre-composed with the conjugation map $z \mapsto \bar{z}$.
- In higher-dimensions, a description as a sphere of positive null-lines and the special Lorentizian group exists in the Lorentzian space $\mathbb{R}^{1,n+1}$.

3.2.5.1 *Poincare extensions*

We can identify E^{n-1} with $E^{n-1} \times \{O\}$ in E^n and extend each Mobius transformation of $\hat{E}^{n-1}$ to $\hat{E}^n$ that preserves the upper half space U: We extend reflections and inversions in the obvious way: by extending a reflection in E^{n-1} about a hyperplane to a reflection in E^n about a hyperplane containing the hyperplane and perpendicular to E^{n-1}, and extending the inversion in E^{n-1} about a sphere of radius r with center $x \in E^{n-1}$ to the inversion in E^n with the same radius and center.

Each Mobius transformation m of $\hat{E}^{n-1}$ is a compositon of reflections and inversions, say $r_1 r_2 ... r_n$. Denoting $\hat{r}_i$ the extension. Let the extension $\hat{m}$ of m be given by $\hat{r}_1 \hat{r}_2 ... \hat{r}_n$.

- The Mobius transformation of $\hat{E}^n$ that preserves the open upper half spaces are exactly the extensions of the Mobius transformations of $\hat{E}^{n-1}$. Therefore, $M(U^n) = M(\hat{E}^{n-1})$.
- We can put the pair $(U^n, \hat{E}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobius transformation η of $\hat{E}^n$. Thus, $M(U^n)$ is isomorphic to $M(\mathbf{S}^{n-1})$ for the boundary sphere by a conjugation by η.

3.2.6 *Hyperbolic geometry*

A hyperbolic space H^n is defined as a Riemannian manifold of constant curvature equal to -1. Such a space cannot be realized as a submanifold in a Euclidean space of even very large dimensions. But it is realized as a "sphere" in a Lorentzian space. A *Lorentzian space* is the vector space $\mathbb{R}^{1+n}$ with an inner product

$$x \cdot y = -x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n y_n.$$

We will denote it by $\mathbb{R}^{1,n}$.

- A Lorentzian norm $||x|| = (x \cdot y)^{1/2}$ is a positive, zero, or positive imaginary number. The vector is said to be *space-like*, *null*, or *time-like* depending on its norm being positive, zero, or positive imaginary number.
- The *null* vectors form a light cone divided into positive, negative cone, and 0.
- The subspace of time-like vectors also has two components where $x_0 > 0$ and $x_0 < 0$ respectively. A time-like vector is also *positive* or *negative* depending on which component it lies in.
- Ordinary notions such as orthogonality can be defined by Lorentzian inner product. A basis is *orthornormal* if its vectors have norms of 1 or i and mutually orthogonal.
- A subspace of $\mathbb{R}^{1,n}$ is either *space-like* where all vectors in it are space-like is *null* where at least one nonzero-vector is null, or finally *time-like* where at least one vector is time-like: This can be seen by looking at the restriction of the Lorentz inner product on subspaces where it can be either positive-definite, semi-definite, or definite with at least one vector with imaginary norm.
- A pair of space-like vectors v and w spanning a space-like subspace have an angle between them given by the formula $\cos\theta = v\dot{w}/||v||||w||$. This can be generalized to the situations where they do not span a space-like subspace and span a null-space or time-like subspaces. (For details, see Ratcliffe [**(author?)** (Ratcliffe)]).

3.2.6.1 *Lorentz group*

A *Lorentzian transformation* is a linear map preserving the inner-product. A *Lorentzian matrix* is a matrix corresponding to Lorentzian transformation under a standard coordinate system. For J the diagonal matrix with entries $-1, 1, ..., 1$, $A^t J A = J$ if and only if A is a Lorentzian matrix.

The set of Lorentzian transformations form a Lie group $O(1,n)$ given by $\{A \in GL(n, \mathbb{R}^{1+n} | A^t J A = J\}$, which is a subgroup of $GL(n, \mathbb{R}^{n+1})$. A Lorentzian transformation sends time-like vectors to time-like vectors. Thus, by continuity, it either preserves both components of the subspace of positive time-like vectors or switches the components. It is either *positive* or *negative* if it sends positive time-like vectors to positive time-like ones or negative time-like ones. The set of positive Lorentzian transformations form a Lie subgroup $PO(1,n)$.

We note here that $PO(1,n)$ can be considered a subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$ simply since the quotient map $GL(n+1,\mathbb{R}) \to \mathrm{PGL}(n+1,\mathbb{R})$ maps the subgroup diffeomorphic to its image subgroup. Hence, there is an inclusion map $PO(1,n) \to \mathrm{PGL}(n+1,\mathbb{R})$.

3.2.6.2 *Hyperbolic space*

Given two positive time-like vectors, the subspace spanned by them is time-like and the Lorentzian inner product restricts to an inner product of signature $-1, 1$. Using a new coordinate system s, t, the inner product becomes $-s^2 + t^2$. Since the absolute values of second components of the two vectors are larger than those of the first components, the inner-product of the two vectors is a negative number, Their norms are positive imaginary numbers, and the absolute value of the inner-product is greater than the product of the absolute values of their norms as can be verified by simple computations. Therefore, there is a *time-like angle* $\eta(x,y)$ defined by

$$x \cdot y = ||x||||y||cosh\eta(x,y).$$

A hyperbolic space is an upper component of the submanifold defined by $||x||^2 = -1$ or $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$. This is a subset of a positive cone. Topologically, it is homeomorphic to $\mathbb{R}^n$ since one can realize it as a graph of the function. `http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/mnkw/`

Need to do something about the internet stuff....

One induces a metric from the Lorentzian space which is positive definite: for two tangent vectors x, y to the hyperboloid, we define $x \cdot y$ by the Lorentzian inner product. Since the tangent vectors at a point u of the hyperbolic is orthogonal to u, the tangent space is space-like and the norms are always positive. This gives us a Riemannian metric of constant curvature -1. (The computation of curvature is very similar to the computations for the sphere.)

A *hyperbolic line* is an intersection of H^n with a time-like two-dimensional vector subspace. A *triangle* is given by three segments meeting at three vertices. Denote the vertices by A, B, and C and the opposite segments by a, b, and c. By denote thing their angles and lengths again by A, B, C, a, b, and c respectively. We obtain

- Hyperbolic law of sines:
$$\sin A/\sinh a = \sin B/\sinh b = \sin C/\sinh c$$
- Hyperbolic law of cosines:
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$$
$$\cosh c = (\cosh A \cosh B + \cos C)/\sinh A \sinh B$$

One can assign any interior angles to a hyperbolic triangle as long as the sum is less than π. One can assign any lengths to a hyperbolic triangle as long as the lengths satisfy the triangle inequality.

A picture needed

We note that the triangle formula can be generalized to formula for quadrilateral, pentagon, hexagon with some right angles. Basic philosophy here is that one can push the vertex outside and the angle becomes distances between lines. (See Ratcliffe, `http://online.redwoods.cc.ca.us/instruct/darnold/staffdev/Assignments/sinandcos.pdf`)

Since $PO(1, n)$ includes $O(n, \mathbb{R})$ acting on the subspace given by $x_0 = 0$ and $PO(1, 1)$ acting transitively on the hyperbolic line through e_0 and e_1, $PO(1, n)$ acts transitively on H^n. Given any isometry k, we can find an element $g \in PO(1, n)$ so that $g \circ k$ fixes e_0 and the tangent space at e_0. By analyticity of the isometry group, it follows that $k = g^{-1}$. Therefore, the Lie group $PO(1, n)$ is the isometry group of H^n and acts faithfully and transitively.

3.2.7 *Beltrami-Klein models of hyperbolic geometry*

The hyperboloid model is a bit complicated in that we have to see one-dimension higher to realize its meaning. We will give more intrinsic definitions which are obtainable from the hyperboloid model easily.

Beltrami-Klein model is directly obtained from the hyperboloid model. Recall that an affine patch $\mathbb{R}^n$ in $\mathbb{R}P^n$ is identifiable with a complement of a subspace. A standard one is given by $x_0 \neq 0$. The standard affine patch has coordinate system $x_1, ..., x_n$. There is an imbedding from H^n onto an open unit ball B in the standard affine patch $\mathbb{R}^n$ of $\mathbb{R}P^n$:
$$[x_0, x_1, x_2, ..., x_n] \to (x_1/x_0, x_2/x_0, ..., x_n/x_0)$$

induced from a standard radial projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$.

We regard B as a ball of radius 1 with center at O in $\mathbb{R}^n$. The hyperboloid has a distance metric induced from the Riemannian metric. By the projection, we obtain a distance metric d_k on B. We compute that $d_k(P,Q) = 1/2 \log |(AB, PQ)|$ where A, P, Q, B are on a segment with endpoints A, B and

$$(AB, PQ) = \left| \frac{AP}{BP} \frac{BQ}{AQ} \right| :$$

We can verify this as follows: The metric is induced on B. This is precisely the metric given by the log of the cross ratio. Note that $\lambda(t) = (\cosh t, \sinh t, 0, ..., 0)$ define a unit speed geodesic in H^n. Under the Riemannian metric, we have $d(e_1, (\cosh t, \sinh t, 0, ..., 0)) = t$ for t positive. Under d_k, we obtain the same. Since any geodesic segment of same length is congruent under the isometry, we see that the two metrics coincide.

The isometry group $PO(1,n)$ also maps injectively to a subgroup of $PGL(n+1, \mathbb{R})$ that preserves B. Since the isometry corresponds to a linear map in $\mathbb{R}^{1+n}$ and it preserve H^n, it follows that an isometry corresponds to a projective automorphism of B. Converesely, we see that a projective automorphism of B preserves d_k because it preserves the cross-ratios and hence, it must come from the isometry. The projective automorphism group of B is precisely $PO(1,n)$.

(See `http://www.math.uncc.edu/~droyster/math3181/notes/hyprgeom/node57.html`)

- Beltrami-Klein model is nice because you can see outside in $\mathbb{R}P^n$. The outside is the anti-de Sitter space `http://en.wikipedia.org/wiki/Anti_de_Sitter_space` We can treat points outside and inside together.
- Each line (hyperplane) in the model is dual to a point outside. (i.e., orthogonal by the Lorentzian inner-product) A point in the model is dual to a hyperplane outside. Infact any subspace of dimenstion i is dual to a subspace of dimension $n - i - 1$ by orthogonality.
- For $n = 2$, the duality of a line is given by taking tangent lines to the disk at the endpoints and taking the intersection.
- The distance between two hyperplanes can be obtained by two dual points. The two dual points span an orthogonal plane to the both hyperperplanes and hence provide a shortest geodesic.

3.2.7.1 *The conformal ball model (Poincare ball model)*

The stereo-graphic projection H^n to the subspace P in $\mathbb{R}^{1+n}$ given by $x_0 = 0$ from the point $(-1, 0, ..., 0)$.

The formula for the map $\kappa : H^n \to P$ is given by

$$\kappa(x) = \left(\frac{y_1}{1+y_0}, ..., \frac{y_n}{1+y_0}\right),$$

where the image lies in an open ball of radius 1 with center O in P. The inverse is given by

$$\zeta(x) = \left(\frac{1+|x|^2}{1-|x|^2}, \frac{2x_1}{1-|x|^2}, ..., \frac{2x_n}{1-|x|^2},\right).$$

Since this is a diffeomorphism, B has an induced Riemannian metric of constant curvature -1. We show

$$\cosh d_B(x, y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)},$$

and inversions acting on B preserves the metric. Thus, the group of Mobius transformations of B preserve metric. The corresponding Riemannian metric is $g_{ij} = 2\delta_{ij}/(1-|x|^2)^2$. It follows that the group of Mobius transformations acting on B is precisely the isometry group of B. Thus, $Isom(B) = M(\mathbf{S}^{n-1})$. Geodesics would be lines through O and arcs on circles perpendicular to the sphere of radius 1.

3.2.7.2 *The upper-half space model.*

Let U be the upper half-space in $\mathbb{R}^n$. Then U is homeomorphic to an open ball in the compactification $\hat{\mathbb{R}}^n = \mathbf{S}^n$. Since B is an open ball, we can find a Mobius transformation sending B to U by a reflection. Now we put B to U by a Mobius transformation. This gives a Riemannian metric constant curvature -1 on U.

We have by computations $\cosh d_U(x, y) = 1+|x-y|^2/2x_n y_n$ and the Riemannian metric is given by $g_{ij} = \delta_{ij}/x_n^2$. Then $I(U) = M(U) = M(E^{n-1})$. Geodesics would be arcs on lines or circles perpendicular to E^{n-1}.

Since $\hat{E}^1$ is a circle and $\hat{E}^2$ is the complex sphere, we obtain $Isom^+(B^2) = PSL(2, \mathbb{R})$ and $Isom^+(B^3) = PSL(2, \mathbb{C})$ respectively. In this model, it is easier to classify isometries.

- Orientation-preserving isometries of hyperbolic plane can have at most one fixed point. An *elliptic* isometry is one fixing a unique point. A *hyperbolic* isometry is one preserving a unique line. The remaining one is a *parabolic* isometry or is the idenity map. The elliptic, hyperbolic, and parabolic isometries are ones conjugate to

$$z \mapsto e^{i\theta}z, z \mapsto az, a \neq 1, a \in \mathbb{R}^+, z \mapsto z+1$$

in the Mobius group.

- Orientation-preserving isometries of a hyperbolic space is classified as loxodromic, hyperbolic, elliptic, or parabolic. Up to conjugations, they are represented as Mobius transformations which has forms
  - $z \mapsto \alpha z, Im\alpha \neq 0, |\alpha| \neq 1.$
  - $z \mapsto az, a \neq 1, a \in \mathbb{R}^+.$
  - $z \mapsto e^{i\theta}z, \theta \neq 0.$
  - $z \mapsto z+1.$

3.3 Discrete group actions

Here, we will let X be generally a manifold with some Lie group G acting on it transitively. In order for most of the developed theory to work, we need that X be a sphere $\mathbf{S}^n$ or $\mathbb{R}P^n$ with Lie groups such as $O(n+1,\mathbb{R})$, $U(n)$, the Mobius group, or $\mathbb{R}^n$ with $O(n)\cdot\mathbb{R}^n$ or $Aff(\mathbb{R}^n) = GL(n,\mathbb{R})\cdot\mathbb{R}^n$ acting, or H^n with $PO(1,n)$ acting on it. Sometimes, we cannot let X be a symmetric spaces with its isometry group even or a complex hyperbolic space. The reason is that there seems to be no good notion of m-planes, m-dimensional subspaces with nice intersection properties, exists. It is a hope of geometric topologists that we can overcome these difficulties.

We will present facts for X that will be useful in many cases with some additional assumptions on X. However, the reader may wish to see X as one of the above. These will be most sufficient.

Mention incidence geometry and Klein geometry: Do some research????. Millman's book... This should be earlier.... Should write some of this....???

3.3.1 *Discrete groups and discrete group actions*

Let X be a manifold. A *discrete group* is a group with a discrete topology. (Usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group. We have many notions of a group action $\Gamma \times X \to X$ which induces a homomorphism $\Gamma \to Diff(X)$ where $Diff(X)$ denotes the group of diffeomorphisms of X with the C_1-topology:

- The action is *effective* if an element g of Γ correspond to I_X if and only if g is the identity in Γ. The action is *free* if an element g fixes a point of X if and only if g is the identity in Γ.

- The action is *discrete* if Γ is discrete in the group of homeomorphisms of X with compact open topology. (We used the fact that $Diff(X)$ is a subgroup of the group of homeomorphisms.)
- The action has *discrete orbits* if every x has a neighborhood U so that the orbit points in U is finite.
- The action is *wandering* if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U) \cap U \neq \emptyset$ is finite.
- The action is *properly discontinuous* if for every compact subset K the set of γ such that $K \cap \gamma(K) \neq \emptyset$ is finite.

We can show that the conditions of discrete action, discrete orbit action, wandering action, and properly discontinuous are strictly stronger according to the order presented here as long as X is a manifold. The proof of this fact without the strictness is not very involved by showing that the later condition implies the given condition.

- If the action is wandering and free, then the action gives manifold quotient which is possibly non-Hausdorff.
- The action of Γ is free and properly discontinuous if and only if X/Γ is a manifold quotient (Hausdorff) and $X \to X/\Gamma$ is a covering map.
- Suppose that Γ a discrete subgroup of a Lie group G acting on X with compact stabilizer. Then X has G-invariant Riemannian metric. Any (X, G)-manifold now has induced Riemannian metric. Suppose that Γ acts properly discontinuously on X. Let us call this *standard discrete action.*
- A *complete* (X, G) *manifold* is one isomorphic to X/Γ. (Notions of completeness agree with that of the induced Riemannian metric for G acting with compact stabilizers. Hence, this is a natural generalization.)
- Suppose X is simply-connected. Given a manifold M the set of complete (X, G)-structures on M up to (X, G)-isotopies are in one-to-one correspondence with the discrete faithful representations of $\pi(M) \to G$ up to conjugations.

We remark that if we allow G to act on X without the compact stabilizer condition, then we call this *standard flexible type* action.

As examples, we give:

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. (free, properly discontinuous.)
- The modular group $PSL(2, \mathbb{Z})$ the group of Mobius transformations or isometries of hyperbolic plane given by $z \mapsto \frac{az+b}{cz+d}$ for integer a, b, c, d and $ad - bc = 1$. `http://en.wikipedia.org/wiki/Modular_group`. This is not a free action but a properly discontinuous action as the action is a standard discrete one.

3.3.1.1 *Convex polyhedrons*

Suppose that X is a space where a Lie group G acts effectively and transitively. Furthermore, suppose X has notions of m-planes. An *m-plane* is an element of a collection of submanifolds of X of dimension m so that given generic $m + 1$ point, there exists a unique one containing them. We require also that every 1-plane contains geodesic between any two points in it. Obviously, we assume that elements of G sends m-planes to m-planes. (For complex hyperbolic spaces, such notion seemed to be absent.)

We also need to assume that X satisifes the increasing property that given an m-plane and if the generic $m + 1$-points in it, lies in an n-plane for $n \geq m$, then the entire m-plane lies in the n-plane. Check Kapovich

For example, any geometry with models in $\mathbb{R}P^n$ and G a subgroup of $\mathrm{PGL}(n + 1, \mathbb{R})$ has a notion of m-planes. Thus, hyperbolic, euclidean, spherical, and projective geometries has notions of m-planes. Conformal geometry may not have such notions since generic pair of points have infinitely many circles through them.

A *convex subset* of X is a subset such that for any pair of points, there is a unique geodesic segment between them and it is in the subset. For example, a pair of antipodal point in $\mathbf{S}^n$ is convex.

Assume that X is either $\mathbf{S}^n$, $\mathbb{R}^n$, H^n, or $\mathbb{R}P^n$ with Lie groups acting on X. Let us state some facts about convex sets:

- The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in X.
- The interior C^o, the boundary ∂C are defined in $\hat{C}$.
- The closure of C is in $\hat{C}$. The interior and closures are convex. They are homeomorphic to an open ball and a contractible domain of dimension equal to that of $\hat{C}$ respectively.
- A side C is a nonempty maximal convex subset of ∂C.

- A convex polyhedron is a nonempty closed convex subset such that the set of sides is locally finite in X.

3.3.1.2 *Convex polytopes*

Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.

- A *convex polytope* in $B = H^n$ is a convex polyhedron with finitely many vertices and is the convex hull of its vertices in $B = H^n$.
- A polyhedron P in $B = H^n$ is a *generalized convex polytope* if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices.
- For $X = \mathbb{R}P^n$ or $\mathbf{S}^n$, a *convex polytope* is given as a convex polyhedron in an affine patch or an open hemisphere with finitely many vertices and is a convex hull of its vertices.
- In general, for X with m-planes, we can define a *convex polytope* as above.

A compact simplex which convex hull of $n + 1$ points in $B = H^n$ is an example of a convex polytope.

Take an origin in B, and its tangent space $T_O B$. Start from the origin O in $T_O B$ expand the infinitesimal euclidean polytope from an interior point radially. That is a map sending $x \to sx$ for $s > 0$ and x is the coordinate vector of an affine patch using in fact any vector coordinates. Now map the vertices of the convex polytope by an exponential map to B. The convex hull of the vertices is a convex polytope. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes.

3.3.1.3 *The fundamental domains of discrete group action*

Let X be $\mathbf{S}^n, E^n$ or H^n or more generalty a geometrical space with m-planes. Let Γ be a group acting on X. A *fundamental domain* for Γ is an open domain F so that $\{gF|g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X. The fundamental domain is *locally finite* if the above closures are locally finite.

Suppose that X is either a hyperbolic, euclidean, or spherical space. Then *Dirichlet domain* for $u \in X$ is the intersection of all $H_g(u) = \{x \in X|d(x, u) < d(x, gu)\}$. Then $D(u)$ is a convex fundamental polyhedron. If

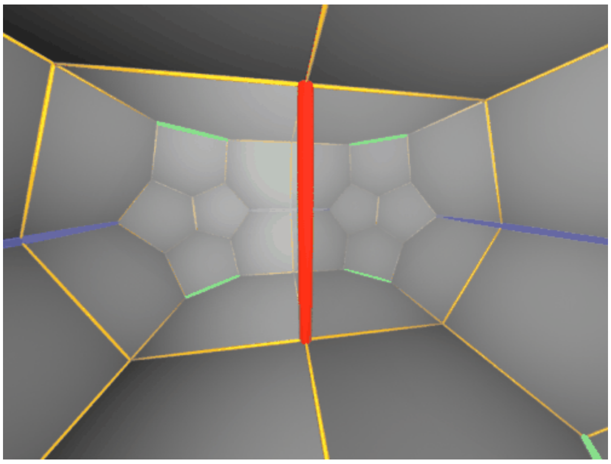


Fig. 3.1 Regular icosahedron with all edge angles $\pi/2$ as seen from inside (Geometry center).

X/Γ is compact, and Γ acts discretely and properly discontinously, $D(u)$ is a convex polytope. (If X is some other types of geometries, this is somewhat only vaguely understood.)

The regular octahedron example of hyperbolic surface of genus 2 is an example of a Dirichlet domain with the origin as u. (See Figure **??**.)

3.3.1.4 *Side pairings and Poincare fundamental polyhedron theorem*

A *tessellation* of X is a locally-finite collection of polyhedra covering X with mutually disjoint interiors.

Convex fundamental polyhedron provides examples of exact tessellations. For such a convex fundamental polyhedron P, X is a union $\bigcup_{g\in\Gamma} g(P)$.

If P is an exact convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by $\Phi = \{g \in \Gamma | P \cap g(P)$ is a side of $P\}$: To see this, let g be an element of Γ, and let us choose a frame at a point of P and consider its image in $g(P)$. Then we choose a path of frames from the intial from to the terminal frame. We perturb the path so that it meets only the interiors of the sides of the tessellating polyhedrons. Each time the path crosses a side S, we take the side-pairing g_S obtained as below. Then multiplying all such side-pairings in the reverse order to what occured, we obtain an element $g' \in \Gamma$ so that $g'(P) = g(P)$

as hg_Sh^{-1} moves $h(P)$ to the image of P adjacent in the side $h(S)$ for every $h \in \Gamma$. Since P is a fundamental domain, $g^{-1}g'$ is the identity element of Γ.

- Given a side S of an exact convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}(P)$.
- Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- $[x]$ is finite and $[x] = P \cap \Gamma$.

- Cycle relations:
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S'_1. Let S_2 be the side adjacent to S'_1 so that $g_{S_1}(S'_1 \cap S_2) = R$.
  - Let S_{i+1} be the side of P adjacent to S'_i such that $g_{S_i}(S'_i \cap S_{i+1}) = S'_{i-1} \cap S_i$.
- Then we obtain
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S'_i, S_{i+1}) = 2\pi/k$.
  - $g_{S_1}g_{S_2}....g_{S_l}$ has order k.
- The period l is the number of sides of codimension one coming into a given side R of codimension two in X/Γ.

We comment that the angle condition is equivalent to the order condition below. If X does not have a G-invariant metric, we can only state the order condition. Thus, if Γ has a convex fundamental polytope, Γ is finitely presented.

The Poincare fundamental polyhedron theorem is the converse. We claim that the theorem holds for geometries (X, G) with notions of m-planes. (See Kapovich P. 80–84):

Check Kapovich more...

Theorem 3.3.1. *Let (X, G) be a geometry of notions of m-planes and convexity. Given a convex polyhedron P in X with side-pairing isometries satisfying the above relations, then P is the fundamental domain for the discrete subgroup of G generated by the side-pairing isometries.*

If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds. The results are always complete. (See

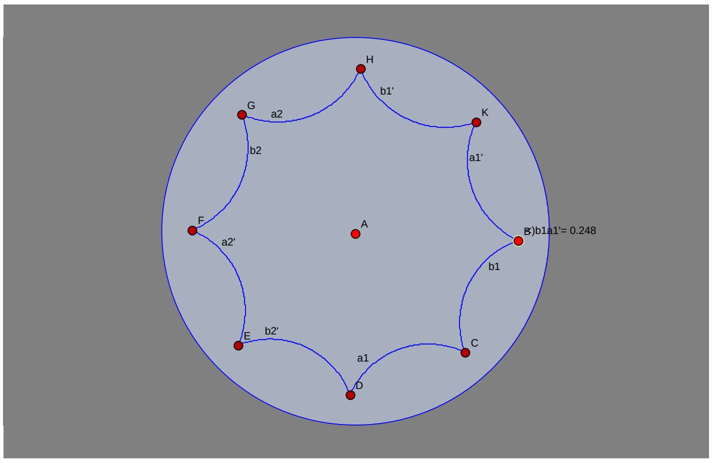


Fig. 3.2 Example: the octahedron in the hyperbolic plane giving genus 2-surface. There are the cycle $(a1, D), (a1', K), (b1', K), (b1, B), (a1', B), (a1, C), (b1, C)$, the cycle $(b1', H), (a2, H), (a2', E), (b2', E), (b2, F), (a2', F), (a2, G)$, and the cycle $(b2, G), (b2', D), (a1, D), (a1', K), ...$

Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html` for an examples of hyperbolic or spherical manifold as seen from "inside".)

We will be particularly interested in reflection groups. Suppose that X has notions of angles between m-planes. A *discrete reflection group* is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Then all the dihedral angles are submultiples of π. Then the side pairing such that each face is glued to itself by a reflection satisfies the Poincare fundamental theorem.

The reflection group has presentation $\{S_i : (S_iS_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$. which are examples of Coxeter groups.

The triangle groups are examples of discrete reflection groups.

- Find a triangle in X with angles submultiples of π. This exists always for $X = \mathbf{S}^2, E^2$, or H^2.
- We divide into three cases $\pi/a+\pi/b+\pi/c > 0, = 0, < 0$. The triangles are then spherical, euclidean, or hyperbolic ones respectively. They exist and are uniquely determined up to isometry.
  - > 0 cases: $(2, 2, c), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ corresponding to dihedral group of order $2c$, a tetrahedral group, octahedral group, and icosahedral group.
  - $= 0$ cases: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$. The reflections generate

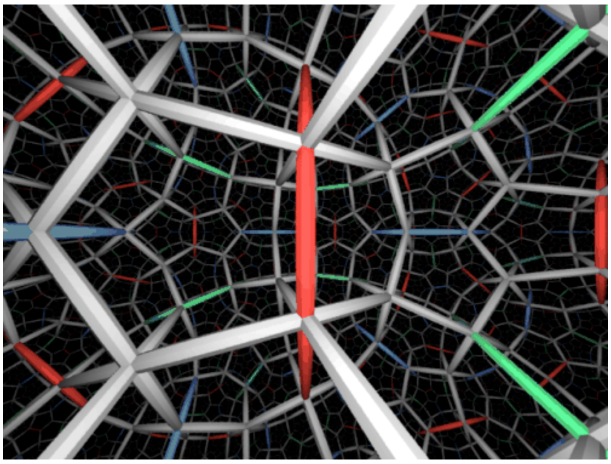


Fig. 3.3 The icosahedral reflection group as seen by an insider: One has a regular icosahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group. From Geometry center

the corresponding wall paper group.
- < 0 cases: Infinitely many hyperbolic tessellation groups. See `http://egl.math.umd.edu/software.html`.

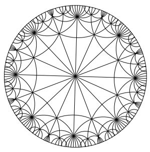


Fig. 3.4 $(2, 4, 8)$-triangle group.

Can this be made bigger?

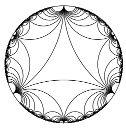


Fig. 3.5 The ideal example.

3.3.1.5 *Higher-dimensional examples*

To construct a 3-dimensional examples, obtain a Euclidean regular dodecahedron and expand it and decrease the dihedral angles until we achieve that all dihedral angles are $\pi/3$ and then to dihedral angles $\pi/2$. There are nice pictures of these in Geometry Center archives.

One can also achieve Regular octahedron with angles $\pi/2$. These are ideal polytope examples.

Higher-dimensional examples were analyzed by Vinberg and so on. For example, there are no hyperbolic reflection group of compact type above dimension ≥ 30.

3.3.1.6 *Crystallographic groups*

A *crystallographic group* is a discrete group of the rigid motions on $\mathbb{R}^n$ whose quotient space is compact.

Bieberbach theorem states that

Theorem 3.3.2.

- *A group is isomorphic to a crystallographic group of* $\mathbb{R}^n$ *if and only if it contains a subgroup of finite index that is free abelian of rank equal to* n.
- *The crystallographic groups are isomorphic as abstract groups if and only if they are conjugate by an affine transformation.*

Once we have this theorem, then the classification of crystallographic group is reduced to studying the finite group extension of abelian crystallographic groups, which are simple lattices. There are only finitely many crystallographic group for each dimension since once the abelian group action is determined, its symmetry group can only be finitely many. There are 17 wallpaper groups for dimension 2. `http://www.clarku.edu/~djoyce/wallpaper/` and see Kali by Weeks `ttp://www.geometrygames.org/Kali/index.`tml. There are 230 space groups for dimension 3. See Conway, Friedrichs, Huson and Thurston [**(author?)** (Conway)] These groups have extensive applications in molecular chemistry. Further informations: `http://www.ornl.gov/sci/ortep/topology.html`

3.4 Notes

A good introduction to Euclidan, affine, and projective geometry can be found in [**?**)] and some early chapters of [Thurston (11)]. The book [**?**)] gives us extensive descriptions of models of hyperbolic geometry. Discrete group actions and Poincare fundamental polyhedron theorems are described well in [**?**)] and [**(author?)** (Kapovich)].

Chapter 4

Topology of orbifolds

4.1 Introduction

This section begins by reviewing the theory of the compact group actions on manifolds. Then we move onto define orbifold and their maps. We also cover the groupoid definition. Further we cover the covering space theory. We exposed the covering theory using paths following Haefliger. Thus, both concrete and abstract approach covered here. We tried to make the abstract definitions into more concrete form here. In many respect, the abstract definition gives a more accurate sense of what orbifold is. Finally, we discussed the topological operations of cutting and pasting along suborbifolds that can be done on orbifolds.

- Compact group actions
  - Compact group actions
  - Orbit spaces.
  - Tubes and slices.
  - Path-lifting, covering homotopy
  - Locally smooth actions
  - Smooth actions
  - Equivariant triangulations
  - Newman's theorem
- Topology of 2-orbifolds
  - Definitions,
  - Orbifold maps, singular set,
  - Examples
  - Abstract definitions using groupoid.
  - Smooth structures, fiber bundles, and Riemannian metrics

  - Gauss-Bonnet theorem (due to Satake)
  - Smooth 2-orbifolds and triangulations
- Covering spaces
  - Fiber-product approach
  - Path-approach by Haefliger
- Topological operations on 2-orbifolds: constructions and decompositions

4.2 Compact group actions

Although, we only need the result for finite group actions, we will study when G is a compact Lie group.

A group action $G \times X \to X$ with $e(x) = x$ for all x and $gh(x) = g(h(x))$. That is, $G \to Diffeo(X)$ so that the product operation becomes compositions.

An *equivariant* map $\phi : X \to Y$ between G-spaces is a map so that $\phi(g(x)) = g(\phi(x))$. An *isotropy subgroup* $G_x = \{g \in G | g(x) = x\}$. We note that $G_{g(x)} = gG_xg^{-1}$ and $G_x \subset G_{\phi(x)}$ for an equivariant map ϕ.

Theorem 4.2.1. (Tietze-Gleason Theorem) *Let G be a compact group acting on X with a closed invariant set A. Let G also act linearly on $\mathbb{R}^n$. Then any equivariant smooth map $\phi : A \to \mathbb{R}^n$ extends to a smooth $\phi : X \to \mathbb{R}^n$.*

An orbit of a point x of X is $G(x) = \{g(x)|g \in G\}$. Then we see that $G/G_x \to G(x)$ is one-to-one and onto continuous function. Therefore, an orbit type is given by the conjugacy class of G_x in G. The orbit types form a partially ordered set induced by the reversing the inclusion ordering of the conjugacy classes of subgroups of G. Denote by X/G the space of orbits with quotient topology.

For $A \subset X$, define $G(A) = \bigcup_{g\in G} g(A)$ is the *saturation* of A.

- $\pi : X \to X/G$ is an open, closed, and proper map.
- X/G is Hausdorff. (as G is compact.)
- X is compact iff X/G is compact.
- X is locally compact iff X/G is locally compact.

We list some examples:

- Let $X = G \times Y$ and G acts as a product. Then every orbit is homeomorphic to G and the centralizers are all trivial groups.

- For k, q relatively prime, the action of Z_k on the unit sphere S^3 in the complex space $\mathbb{C}^2$ generated by a matrix
$$\begin{bmatrix} e^{2\pi i/k} & 0 \\ 0 & e^{2\pi q i/k} \end{bmatrix}$$
giving us a Lens space.
- We can also consider S^1-actions given by
$$\begin{bmatrix} e^{2\pi k i\theta} & 0 \\ 0 & e^{2\pi q i\theta} \end{bmatrix}$$
Then it has three orbit types.
- Consider in general the action of torus T^n-action on C^n given by
$$(c_1, ..., c_n)(y_1, ..., y_n) = (c_1 y_1, ..., c_n y_n), |c_i| = 1, y_i \in C.$$
Then there is a homeomorphism $h : C^n/T^n \to (R^+)^n$ given by sending
$$(y_1, ..., y_n) \mapsto (|y_1|^2, ..., |y_n|^2).$$
The interiors of sides represent different orbit types.
- Let H be a closed subgroup of Lie group G. Let H act on G by the left action. The left-coset space G/H is the orbit space where G acts on the right also.
- $G/G_x \to G(x)$ is given by $gG_x \mapsto g(x)$ is a homeomorphism if G is compact.
- Twisted product: X a right G-space, Y a left G-space. A left action is given by $g(x, y) = (xg^{-1}, gy)$. The twisted product $X \times_G Y$ is the quotient space.
- Let $p : X \to B$ is a principal bundle with G acting on the left. F a right G-space. Then $F \times_G X$ is the associated bundle.

Example 4.2.2. Let G be the the rotation group $SO(3)$, and let X be the vector space of symmetric matrices of trace 0 (hence orthogonally diagonalizable). Suppose we act by conjugation $g(m) = gmg^{-1}$. By linear algebra, we prove that two symmetric marices are in the same orbit if they have the same eigenvalues with multiplicities. Hence the orbit space is in one-to-one correspondence with of triples (a, b, c) so that $a \geq b \geq c$ and $a + b + c = 0$. The second space is a 2-dimensional cone in $\mathbb{R}^3$. This is homeomorphic to X/G. The isotropy group of a diagonal matrix with all eigenvalues different is the group of diagonal matrices with entries ± 1 which is isomorphic to $Z_2 \oplus Z_2$. The isotropy group of a diagonal matrix with two eigenvalues equal is the group of matrices decomposing into an orthogonal 2×2-matrix and ± 1.

Example 4.2.3. The Conner-Floyd example: There is an action of Z_r for $r = pq$,p, q relatively prime, on an Euclidean space of large dimensions without stationary points. That is the stabilizers of every point is a trivial group. This is accomplished in following steps:

- Find a simplicial action Z_{pq} on $S^3 = S^1 \star S^1$ without stationary points obtained by joining action of Z_p on the first factor circle and Z_q on the second factor circle.
- Find an equivariant simplicial map $h : S^3 \to S^3$ which is homotopically trivial.
- Build the infinite mapping cylinder using h infinitely often which is contactible and imbed it in an Euclidean space of high-dimensions where Z_{pq} acts orthogonally.
- Find the contractible neighborhood. Taking the product with the real line makes it into a Euclidean space.

Define station-ary...

4.2.1 *Tubes and slices*

For a compact group action, we need to establish the notion of tubes and slices. These are modeled on twisted product action: Let G be a compact subgroup, X a right G-space, and Y a left G-space. Then $X \times_G Y$ is defined as the quotient space of $X \times Y$ where $[xg, y] \sim [x, gy]$ for $g \in G$. Let H be a closed subgroup of G. $G \times_H Y$ is a left G-space by the action $g[g', a] = [gg', a]$ as this sends equivalence classes to themselves. The inclusion $A \to G \times_H A$ induces a homeomorphism $A/H \to (G \times_H A)/G$.

The isotropy subgroup at $[e, a]$ is computed as follows: $[e, a] = g[e, a] = [g, a] = [h^{-1}, h(a)]$. Thus, $G_{[e,a]} = H_a$.

As an example, let $G = S^1$ and A be the unit-disk and $H = \mathbb{Z}_3$ generated by $e^{2\pi/3}$. G and H acts in a standard way in A. Then consider $G \times_H A$. The result is homeomorphic to a solid torus fibered with circles. Each noncentral circle goes around the solid torus three times and the central circle goes around once.

Let X be a G-space and P an orbit of type G/H. A *tube about* the orbit P is a G-equivariant imbedding $G \times_H A \to X$ onto an open neighborhood of P where A is a some space where H acts on so that

- Every orbit passes the image of $e \times A$.
- P equals $G(x)$ for $x = [e, a]$ where a is the stationary point of H in A.

- In general $G_x = H_a \subset H$ for $x = [e, a]$.

Let $x \in X$. Suppose S is a set containing x such that $G_x(S) = S$,i.e., the stablizer of x acts on S. Then S is said to be a *slice* if $G \times_{G_x} S \to X$ so that $[g, s] \to g(s)$ is a tube about G_x.

It is easy to see that S is a slice if and only if S is the image of $e \times A$ for some tube.

Let $x \in S$ and $H = G_x$. Then the following are equivalent:

- There is a tube $\phi : G \times_H A \to X$ about $G(x)$ such that $\phi([e, A]) = S$.
- S is a slice at x.
- $G(S)$ is an open neighborhood of $G(x)$ and there is an equivariant retraction $f : G(S) \to G(x)$ with $f^{-1}(x) = S$.

Theorem 4.2.4. (Mostow) *Let X be a completely regular G-space. There is a tube about any orbit of a complete regular G-space with G compact.*

Proof. Let x_0 have an isotropy group H in G. Find an orthogonal representation of G in $\mathbb{R}^n$ with a point v_0 whose isotropy group is H, which always exists by a compact group representation theory. There is an equivalence of orbits $G(x_0)$ and $G(v_0)$. We extend this to a neighborhood smoothly. For $\mathbb{R}^n$, we can find the equivariant retraction. Transfer this on X. $\square$

If G is a finite group acting on a manifold, then a tube is a union of disjoint open sets and a slice is an open subset where G_x acts on.

Theorem 4.2.5. *(Path-lifting and covering homotopy theorem)*

- *Let X be a G-space, G a compact Lie group, and $f : I \to X/G$ any path. Then there exists a lifting $f' : I \to X$ so that $\pi \circ f' = f$.*
- *Let $f : X \to Y$ be an equivariant map. Let $f' : X/G \to Y/G$ be an induced map. Let $F' : X/G \times I \to Y/G$ be a homotopy preserving orbit types that starts at f'. Then there is an equivariant $F : X \times I \to Y$ lifting F' starting at f.*
- *If G is finite and X a smooth manifold with a smooth G-action and if the functions are smooth, then the lifts can be chosen to be also smooth.*

4.2.1.1 *Locally smooth actions*

Let M be a G-space with G a compact Lie group, and let P be an orbit of type G/H. and V a vector space where H acts orthogonally. Then a *linear* tube in M is a tube of the form $\phi : G \times_H V \to M$.

Let S be a slice. S is a *linear slice* if $G \times_{G_x} S \to M$ given by $[g, s] \to g(s)$ is equivalent to a linear tube. (If G_x-space S is equivalent to the orthogonal G_x-space.)

If there is a linear tube about each orbit, then M is said to be *locally smooth.*

Lemma 4.2.6. *There exists a maximum orbit type G/H for G. (That is, H is conjugate to a subgroup of each isotropy group.)*

Proof. In each tube, there is a maximal orbit type in it and we find the maximal orbit in it has to be dense and open. For intersection of two tubes, the maximal orbit has to be dense and open in both tubes. Thus, the maximal orbit of a tube is of the maximal orbit type in M. □

The maximal orbits so obtained in a tube are called *principal orbits.* If M is a smooth manifold and compact Lie G acts smoothly, this is true.

4.2.1.2 *Manifolds as quotient spaces.*

Finally, we wish to understand about the quotient spaces.

Theorem 4.2.7. *Let M be a smooth manifold, and G a compact Lie group acting smoothly on M. If G is finite, then this is equivalent to the fact that each $i_g : M \to M$ given by $x \mapsto g(x)$ is a diffeomorphism. Let n be the dimension of M and d the dimension of the maximal orbit. Then $M^* = M/G$ is a manifold with boundary if $n - d \leq 2$.*

Proof. Let $k = n - d$ be the codimension of the principal orbits. Consider a linear tube $G \times_K V$. The orbit space $(G \times_K V)^* \cong V^*$. Let S be the unit sphere in V. Then V^* is a cone over S^*. We have that $\dim M^* = \dim V^* = \dim S^* + 1$.

If $k = 0$, then M^* is discrete. If M is a sphere, then M^* is one or two points.

If $k = 1$, then M^* is locally a cone over one or two points. Hence M^* is a 1-manifold. If $k = 2$, then M^* is locally a cone over an arc or a circle. (as S^* is a 1-manifold by the previous step.) □

Example: $\mathbb{Z}_2$ action on $\mathbb{R}^3$ generated by the antipodal map. The result is not a manifold.

4.2.1.3 *Smooth actions are locally smooth*

Recall smooth actions. Let G be a compact Lie group acting smoothly on M. Then there exists an invariant Riemannian metric on M. Then $G(x)$ is a smooth manifold where $G/G_x \to G(x)$ is a diffeomorphism. Recall the exponential map for Riemannian manifolds: For any vector $X \in T_pM$, there is a unique geodesic γ_X with tangent vector at p equal to X. The exponential map $\exp : T_pM \to M$ is defined by $X \mapsto \gamma_X(1)$.

If A is an invariant smooth submanifold, then A has an open invariant tubular neighborhood. This follows by using the normal bundle to A and the exponential map restricted to the normal bundle N_A. Then this map is a local diffeomorphism in a neighborhood N of A in N_A. By taking the same radius open balls in the normal bundle, we obtain the invariant tubular neighborhood as its image.

Prop 4.1. The smooth action of a compact Lie group is locally smooth.

Proof. Use the fact that orbits are smooth submanifolds and the above statements. □

We will need the following facts:

- The subspace $M_{(H)}$ of same orbit type G/H is a smooth locally-closed submanifold of M.
- A a closed invariant submanifold. Then any two open (resp. closed) invariant tubular neighborhoods are equivariantly isotopic.

Theorem 4.2.8. *(Newman's theorem) Let M be a connected topological n-manifold. Then there is a finite open covering $\mathcal{U}$ of the one-point compactification of M such that there is no effective action of a compact Lie group with each orbit contained in some member of $\mathcal{U}$.*

The proof follows from algebraic topology.

Corollary 4.2.9. *If G is a compact Lie group acting effectively on M, then the set of fixed points M^G is nowhere dense.*

4.2.1.4 *Equivariant triangulations*

Sören Illman proved:

Theorem 4.2.10. *Let G be a finite group. Let M be a smooth G-manifold with or without boundary. Then we have:*

- *There exists an equivariant simplicial complex K and a smooth equivariant triangulation $h : K \to M$.*
- *If $h : K \to M$ and $h_1 : L \to M$ are smooth triangulations of M, there exist equivariant subdivisions K' and L' of K and L, respectively, such that K' and L' are G-isomorphic.*

This result was widely used once a proof by Yang (1963) was given. But an error was discovered by Siebenmann (1970) and proved in 1977. I don't think I need this...

4.3 Definition of orbifolds

Let X be a Hausdorff second countable topological space. Let n be fixed. Consider an open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it and a G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$. An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.

- Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
- Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$. Conversely, any $i' : \tilde{U} \to \tilde{V}$ lifting an inclusion $U \to V$ equals $h \circ i$ for $h \in H$. (See Proposition A.1 [**?**)].)

Two charts $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$ are *compatible* if for every $x \in U \cap V$, there is an open neighborhood W of x in $U \cap V$ and a chart $(\tilde{W}, K, \mu)$ such that there are embeddings to $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$. (One can assume W is a component of $U \cap V$.)

If we allow $\tilde{U}$ to be an open subset of the closed upper half space, then the orbifold has *boundary*.

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.

- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set by inclusion relation. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An *orbifold* is a topological space X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point $x \in \tilde{U}$, one can find a finite subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice O_x of coordinate charts since smooth action is locally smooth (linear). We call such a chart (O_x, G_x) a *linear chart.* Therefore, given an orbifold atlas, there is a compatible orbifold atlas consisting of only linear charts.
- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ where $\mathcal{U}$ and $\mathcal{V}$ are maximal charts is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.

4.3.1 *Local group and the singular set*

Let $x \in X$. A *local group* G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.

- This is independently defined up to conjugacy for any choice of y.
- Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

The *singular set* is a set of points where G_x is not trivial. In each chart, the set of fixed points of each subgroup of G_x is a closed submanifold.

Let (O_x, G_x) and (O_y, G_y) be two charts. Subgroups H of G_x and H' of G_y are *strictly topologically conjugate* if there is a chart (U_z, G_z) with morphisms into (O_x, G_x) and (O_y, G_y) in the orbifold atlas so that H and H' correspond to an identical subgroup in G_z. H and H' are *topologically conjugate* if there exists a sequence $H_1 = 0, H_1, \cdots, H_n = H'$ where H_i and H_{i+1} are strictly topologically conjugate.

The subset of the singular set where the conjugacy class of G_x is constant is a relatively closed submanifold. Thus X becomes a stratified smooth topological space where the strata is given by the smooth topological conjugacy classes of subgroups of local groups G_x for $x \in X$.

A *suborbifold* Y of an orbifold X is an imbedded subset such that for each point y in Y and and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G acts on and $G|P$ is the image of the restriction homomorphism of G to P.

Compared to the definition of Adem et al [**?**)], our definition is stronger. The basic reason is so that we can do surgeries along the suborbifolds.

Clearly, manifolds are orbifolds. But as an orbifold, it carries more charts. For example $\mathbb{R}P^n$ will have a chart with $\mathbb{Z}_2$ action on it. By an abuse of notations, a manifold in this paper will mean a manifold with the extended collection of charts as orbifolds. In general, let G be a finite group acting on a manifold M smoothly. Then M/G is a topological space with an orbifold structure with an atlas of charts based on H-invariant open set in M and a subgroup H of G as a model.

Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points.

Let X be a smooth surface. Take a discrete subset. For each point, take a disk neighborhood D with a chart (D', Z_n, q) where D' is a disk and Z_n acts as a rotation with O as a fixed point and $q : D' \to D$ as a cyclic branched covering.

In general, a regular branched covering of a surface by another surface gives us an orbifold structure.

Given a manifold M with boundary. A *doubled* $\hat{M}$ is obtained by $M \times \mathbb{Z}_2/\sim$ where $(x, 0) \sim (y, 1)$ if and only if $x = y \in \partial M$. A $\mathbb{Z}_2$-action $\hat{M}$ is induced by $(x, 0) \mapsto (x, 1)$ and $(x, 1) \mapsto (x, 0)$ for $x \in M$. We can double M as a manifold $\hat{M}$ and obtain $\mathbb{Z}_2$-action. Thus, M can be given an orbifold structure of $\hat{M}/\mathbb{Z}_2$.

We can modify this a little bit. Take a surface and make the boundary be a union of piecewise smooth curves with corners.

- The interiors of some selected arcs are given $\mathbb{Z}_2$ as groups but not all. If the end point of the arc is not in another selected arc, then our model open set is a half open set and $\mathbb{Z}_2$ acts on it as a reflection. This is a *silvering*.
- If two such arcs meet at a point, then the vertex is given a dihedral group as a group.
- Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
- A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

4.3.2 *Triangulation of orbifolds*

In general, a smooth orbifold has a smooth topological stratification and a triangulation so that each open cell is contained in a single strata of same dimension. Smooth topological triangulations satisfying certain weak conditions have a triangulation. One should show that the stratification of orbifolds by orbit types satisfies this condition. Verona [**?**)] provides a complete reference. We will treat this in more detail in Appendix A.

4.3.2.1 *Triangulation of the stratified spaces*

A *manifold M with corner* is a topological manifold with boundary with atlas of charts to $\mathbb{R}^{+,n} = \{(x_1, ..., x_n)|x_1 \geq 0, ..., x_n \geq 0\}$, the boundary of which is a union of open subspaces of dimension i for $i = 0, ..., n-1$. The boundary of M is divided into an open i-dimensional submanifolds of points with neighborhood a neighborhood of a point of one of the i-dimensional submanifolds in $\partial\mathbb{R}^{+,n}$.

A *face* of a topological space A is a closed subset of A with a smooth imbedding $F_B : U_B \to B \times \mathbb{R}^+$ for a neighborhood U_B of B sending B to $B \times \{0\}$. F_B and U_B are said to be the collar and the collar neighborhood. We write $F_B = (p_B, r_B)$.

A Hausdorff, locally compact, paracompact space with countable basis is said to be a *nice* space. Let A be a nice topological space and $X \subset A$ be a locally closed set. A *tube* T_X of X is a neighborhood of X in A with a retraction $\pi_X : T_X \to X$ and a function $\rho_X : T_X \to \mathbb{R}$.

Given two strata X, Y, if $X \subset \mathrm{Cl}(Y)$, then we write $X < Y$. The *dimension of a strata* is a dimension as a manifold. The *depth* of a stratified space is a maximal number of a chain $X_0 < X_1 < \cdots < X_n$ of strata X_i.

Define $T_X^\epsilon = \{a \in T_X | \rho(a) < \epsilon(\pi_X(a))\}$ for a function $\epsilon : X \to \mathbb{R}$. If $X \subset U \subset A$ for an open U, then $T_X^\epsilon \subset U$ for some ϵ. $(\pi_X, \rho_X)|T_X^\epsilon$ is a proper map into $X \times [0, \epsilon) = \{(x, t)|x \in X, 0 \leq t \leq \epsilon(x)\}$.

A *abstract stratification* $\mathcal{A}$ consists of a nice space A and a locally closed subsets A' (strata) of A so that A is a disjoint union of A' with the following properties:

- If $X, Y \in A'$ with $X \cap \mathrm{Cl}(Y) \neq \emptyset$, then $X < Y$.
- Each stratum is a manifold with empty or nonempty boundary
- For any X, X has ϵ_X so that $T_X^\epsilon \cap Y \neq \emptyset$ for $Y \in A'$ implies $X < Y$ and $(\pi_X, \rho_X) : T_X^\epsilon \cap Y \to X \times (0, \epsilon)$ is a submersion.
- For any $X, Y \in A'$, $X \subset \mathrm{Cl}(Y)$, ϵ_X and ϵ_Y satisfy $a \in T_X^\epsilon \cap T_Y^\epsilon$ implies $\pi_Y(a) \in T_X, \pi_X(\pi_Y(a)) = \pi_X(a)$ and $\rho_X(\pi_Y(a)) = \rho_X(a)$.

We now add the notion of faces to a abstract stratification $\mathcal{A}$. In addition to above, we have a family of faces A_i of A called faces with property:

- For each face A_i, there is a collar neighborhood U_{A_i}. For any $X \in A'$, $X \cap U_{A_i}$ is the collar neighborhood of $A_i \cap X$ in X.
- Each stratum $X \in A'$ is a manifold with faces $A_i \cap X$.
- $\pi_X^{-1}(A_i \cap X) = A_i \cap T_{X_i}$ and the collar F_{X_i} is induced from a collar F_{A_i} and $\rho_X = \rho_X \circ p_{A_i}$ in a neighborhood of $A_i \cap X$.

A *relative manifold* (with corners) is a pair of topological spaces $(V, \delta V)$ so that δV is a closed subset of V and $V - \delta V$ is a manifold with corners. A *smooth triangulation* of a relative manifold $(V, \delta V)$ is a map ϕ form a complex K with a subcomplex δK such that $\phi(\delta K) = \delta V$ and $K - \delta K$ is a homeomorphism onto $V - \delta V$ and for any simplex σ, $\sigma - \delta V$ is a smooth imbedding.

A *smooth triangulation* of an abstract stratification $\mathcal{A}$ is a triangulation (K, ϕ) of A satisfying for each stratum X, there is a subcomplex K_X so that $K_X, \phi|K_X$ is a smooth triangulation of $(\mathrm{Cl}(X), \mathrm{Cl}(X) - X)$.

A stratification has a finite depth if the dimension of the stratas are finite and if $X \subset \mathrm{Cl}(Y)$ for a strata, then the dimension of X is strictly less than that of Y.

Theorem 4.3.1. *(Verona) Let $\mathcal{A}$ be an abstract stratification of finite depth. Then there exists a smooth triangulation of $\mathcal{A}$.*

4.3.2.2 *Orbifold as stratified space*

Lemma 4.3.2. *Let V be a euclidean vector space or a half space given by $x_n \geq 0$. Let ∂V denote $x_n = 0$.*

- *The fixed point set F_G of a linear finite group G action is a closed subspace of a vector space.*
- *The subset $F_{G'}$ of points fixed exactly by a subgroup G' of G is a vector subspace with a finite number of closed subspaces removed. $F_{G'}$ is dense open in the subspace of fixed points of G'.*
- *For distinct subgroups G' and G'', $F_{G'}$ and $F_{G''}$ are disjoint.*
- *If $G'' \subset G'$ properly, then $F_{G'}$ is in the closure of $F_{G''}$.*
- *The components of $F_{G'}$ for subgroups G' of G form a stratification of V with faces $F_{G'} \cap \partial V$. and their images under $V \to V/G$ also form a stratification of V/G with faces images of $F_{G'} \cap \partial V$.*

Proof. The first item is clear.

The second follows from the fact that the fixed point set of any subgroup is a subspace. One has to remove subspaces fixed by a larger group from inside. The third item is also clear. For the fourth item, the closure of $F_{G''}$ contains the closure of $F_{G'}$.

For the final item, we prove by induction on the dimension n of V. If $n = 1$, G can only be a reflection group of order two and the statements are clear.

Suppose that the item is true for $n \leq i$. Let V have dimension $i + 1$. If there are subgroups G' with a different fixed set from G, then we are done. If there is such a subgroup G', then the subspace of fixed points of G' is of lower dimension then V. There can be finitely many nontrivial such minimal subgroups. These subspaces meet transversally forming a stratification. The common intersection subspace will have different local group containing the both groups. Each of the subspaces will be stratified by induction.

The map $V \to V/G$ preserves stratification. □

Prop 4.2. The singularity x of an orbifold O with a local group G_x always lies in a submanifold of the local group locally conjuguate to G_x. This forms an abstract stratification of the underlying space of the orbifold O with face ∂O, the boundary of O. Hence, O with the stratification is smooth triangulated.

Proof. First, let G_x be a nontrivial local subgroup. Then the set of points with locally conjugate local group form a locally closed connected manifold by the existence of linear charts and Lemma 4.3.2

Thus, the underlying space X of O is a disjoint union of connected submanifolds determined by the local conjugacy classes of the local groups. Let us call the collection of connected submanifolds $\mathcal{A}$. X satisfies the topological conditions. The set $\mathcal{A}$ form a stratification:

Suppose $X \cap \mathrm{Cl}(Y) \neq \emptyset$ for two strata X, Y. Given the local linear chart for $x \in X$, G_x is the maximal local group in the chart. Then $X \cap U \subset \mathrm{Cl}(Y) \cap U$ for each linear chart neighborhood U of X. Hence $X \subset \mathrm{Cl}(Y)$.

First, we put a Riemannian metric with totally geodesic boundary by Theorem ???. Done this?

To show that these form an abstract stratification, we do induction on the dimension. For $n = 1$, this is clear. Suppose that the abstract stratification property holds for dimension i. Suppose that O has dimension $i + 1$.

Let O^j be the union of j-dimensional strata for $0 \leq j \leq i$.

For each point of O^o, we can find a chart and a equivariant regular neighborhood of it in the chart. We call the union N^o of the image of the regular neighborhoods. Remove N^o from X. Consider $O^1 - N^o$. This is a union of disjoint 1-dimensional strata. For each component, we can cover it by linear charts. The local group actions extend each other in each linear chart. Thus for a component l, there is a manifold U_l with a group G_l acting on it with the set of fixed points l and U_l/G_l maps into X homeomorphically to a neighborhood of l. We can take a G_l-equivariant regular neighborhood V_l in U_l. We call N^1 the union of the images of the regular neighborhoods for each component.

Suppose we defined N^j for a fixed j. Then we define N^{j+1}: We look at all $j+1$-dimensional strata of $X - N^j$. Take a component l. The local group of each points are conjugate and they extend each other. Thus, we form a manifold U_l with a group G_l acting on it. Form a G_l-equvariant regular neighborhood V_l of l in U_l. Again, we let N^{j+1} denote the union of the images of the regular neighborhoods. We can consider each covering manifold as h-dimensional manifold times a $n - h$-dimensional ball with a linear action. (This is verified again by induction.)

After constructing all N^js. We go to the underlying set δX of the orbifold boundary ∂O in X. Then $X - \bigcup_{k=0}^{j} N^k$ is a manifold with boundary $\delta X - \bigcup_{k=0}^{j} N^k$. Thus, we define ρ_X, ϵ_X, and the tubular neighborhoods $T_{\delta X}$ easily.

Now we go to the top dimensional N^j. Then we define ρ_j, ϵ_j, and T_j easily since the j-dimensional strata is a manifold with topological boundary in $X - \bigcup_{k=0}^{j-1} N^k$. Then the result on $X - \bigcup_{k=0}^{j-1} N^k$ is an abstract complex with faces in δX.

Next, we go to N^{j-1} so on. Suppose we constructed for N^{h+1} for some h and defined functions and tubular neighborhoods so that $X - \bigcup_{k=0}^{h} N^k$ is an abstract complex with faces in δX. Now, we go to h-dimensional strata. In $X - \bigcup_{k=0}^{h} N^k$, the h-dimensional strata is a union of h-dimensional manifolds. $N^h - \bigcup_{k=0}^{h} N^k$ is regularly covered by a manifold V_j with finite group actions in each component. $N^h \cap \bigcup_{k=h+1}^{j} N^k$ is an abstract stratification. We consider $N^h \cap \bigcup_{k=h+1}^{j} O^k$. Then $\partial N^h - \bigcup_{k=0}^{h-1} N^k$ is an abstract stratified space. Since each component of N^h is covered by a manifold by a finite group action, we can consider each covering manifold as h-dimensional manifold times a $n-h$-dimensional ball with a linear action and it follows that we can extend the stratified space radially. The extension agrees with X^k for $k \geq h$.

This proves that O has an abstract stratification. Finally, we obtain the smooth triangulation by Theorem 4.3.1. $\square$

4.4 Definition as a groupoid

We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach. However, there are many reasons to learn orbifolds as groupoids since this framework provides us with much more tools and insights from category theory and even from smooth manifold theory in categorical setting. These definitions are mainly introduced to study sheaf theoretic considerations and to pull back bundles and so on.

Here, we will try to minimize the theoretical aspect. In spite of the technical nature, once readers are somewhat aquainted with category theory will recognize these definitions are very concrete. Only the abstract nature of category theory comes when discussing the equivalences of these structures.

4.4.1 *Groupoids*

A *topological groupoid* consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps: the source map $s : G_1 \to G_0$, target map $t : G_1 \to G_0$, an associative composition map $m : G_{1s} \times_t G_1 \to G_1$ a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(y)$ and $gu(x) = g = u(x)g$ and an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.

It will be a convenient to think of these arrows as points as restrictions of smooth maps to points.

A *Lie groupoid* is one where G_0 and G_1 are smooth manifolds.

Let M be a smooth manifold. Let $G_0 = G_1 = M$ and all maps identity, then this is a *unit groupoid.*

As a simple example, let a Lie group K act smoothly on a smooth manifold M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection to M factor and t the action $K \times M \to M$. The unit map is the inclusion map $g \mapsto (e, g)$ for the unit element e of K. The inverse map $K \times M \to K \times M$ is given by $(g, x) \mapsto (g^{-1}, g(x))$.

If K is the trivial group, we obtain the unit Lie groupoid.

- The *isotropy group* at x is the set of all arrows from x to itself.
- A *homomorphism* of Lie groupoids $\phi : H \to G$ is a pair of smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$ commuting with all structure maps.
- The fiber-product: $\phi : H \to G, \psi : K \to G$ the fiber product $H \times_G K$ is the Lie groupoid whose objects are (y, g, z) for $y \in H_0, z \in K_0$, and arrow $\phi(y) \to \psi(z)$ and whose arrows $(y, g, z) \to (y', g', z')$ are pairs (h, k) of arrows $h : y \to y', k : z \to z'$ so that $g'\phi(h) = \psi(h)g$.

An *etale* map of a Lie groupoid is a homomorphism $\phi : G \to H$ so that $\phi_0 : G_0 \to H_0$ is a local homeomorphism. A homomorphism of Lie groupoids $\phi : H \to G$ is an *equivalence* if it is an etale map and

- If ϕ_0 induces an isomorphism of the isotropy group from x to that of $\phi_0(x)$.
- If ϕ induces a bijection of orbit spaces.

If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a *differentiable equivalence* if ϕ_0 is an equivalence and is a local diffeomorphism. This generates an equivalence relation on groupoids.

We can show that two groupoids are equivalent if and only if they are Morita equivalent: i.e., there exists another groupoid and an equivalence map from it to the two groupoids. This essentially means that there are larger groupoid containing both.

4.4.1.1 *A nerve of a groupoid*

Let G be a Lie groupoid. Define

$$G_n = \{(g_1, ..., g_n) | g_i \in G_1, s(g_i) = t(g_{i+1})\}$$

as a fiber product. The face operator $d_i : G_n \to G_{n-1}$ by sending $(g_1, ..., g_n)$ to $(g_1, ..., g_i g_{i+1}, ..., g_n)$. This forms an abstract simplicial manifold. This is said to be the nerve of the groupoid G.

The *classifying space* BG is defined to be the geometric realization as a simplicial complex.

We define the fundamental group π_n of an orbifold X with G as defined as $\pi_n(BG)$.

Later... not defined yet

4.4.2 *An abstract definition*

- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- A groupoid is *foliation* if each isotropy group G_x is discrete.

If G is an etale groupoid, then any arrow $g : x \to y$ in G induces a well-defined germ of a diffeomorphism $\tilde{g} : U_x \to V_y$ for neighborhoods U_x of x and V_y of y, defined as $\tilde{g} = t \circ \hat{g}$, where $\hat{g} : U_x \to G_1$ is a section of the source map $s : G_1 \to G_0$ with $\hat{g}(x) = g$. (By etale property, such sections exist) We call G *effective* (or *reduced*) if the assignment $g \mapsto \tilde{g}$ is faithful; or equivalently, if for each point $x \in G_0$ this map $g \mapsto \tilde{g}$ defines an injective group homomorphism $G_x \to Diff_x(G_0)$.

Theorem 4.4.1. *Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an orbifold.*

We do not prove this theorem; however,

We show below that the orbifold give rise to proper effective etale groupoid.

Example 4.4.2. Let M be a smooth orbifold with an locally finite atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Here U and V could be equal and the identification by X is one given by the local group action on U. Then the space of orbits is X and M_0 and M_1 contains all the information of the atlas. The fact that this is a proper effective etale groupoid follows by checking the above definitions.

Read more here.... Compare with Moer 's definition, It is different from Adem's.. Check this again.....

One complication of using definition like this is that an equivalent groupoid to orbifold groupoid may not be an orbifold groupoid. However, it can be verified that given an equivalence class there exists a unique orbifold isomorphic up to topological orbifold equivalences. (See the book [**?**)]). Moreover, the following groupoid equivalence is a topological orbifold equivalence.

Precise location???

Definition 4.4.3. An *orbifold structure* on a paracompact Hausdorff space X consists of orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. Two orbifold structures $(\mathcal{G}, f)$ and $(\mathcal{H}, g : |\mathcal{H}| \to X)$ are *equivalent* if $\phi : \mathcal{H} \to \mathcal{G}$ is a groupoid equivalence inducing the homeomorphism $|\phi| : |\mathcal{H}| \to X = |\mathcal{G}|$ so that $f \circ |\phi| = g$.

4.4.2.1 *Action of a Lie groupoid*

Let G be an orbifold groupoid. A *left G-space* is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.

- This map is defined on (g, e) with $\pi(e) = s(g)$ and written $\mu(g, e) = g.e$.
- It satisfies the action identity: $\pi(g.e) = t(g)$, $1_x.e = e$, and $g.(h.e) = (gh).e$ for $h : x \to y$ and $g : y \to z$ and $e \in E$ with $\pi(e) = x$.

A *right G-space* is left G^{op}-space obtained by switching the source and target map.

4.5 Differentiable structures on orbifolds

Now, we go back to the original definition of orbifolds using charts.

Suppose we are given smooth structures on each $(\tilde{U}, G, \phi)$, i.e., $\tilde{U}$ is given a smooth structure and G is a smooth action on it. We assume that all embeddings in the atlas is smooth. Then M is given a *smooth* structure.

Given a chart $(\tilde{U}, G, \phi)$, *the space of smooth forms* is the space of smooth forms in $\tilde{U}$ invariant under the G-action. A smooth form on the orbifold is the collection of smooth forms on each of the charts so that under embeddings they correspond.

One can define an integral of smooth singular simplices into charts. This can be extended to any smooth simplex using partition of unity and varicentric subdivisions of the simplex. Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)

- We refine to obtain a cover whose closures are invariant compact subsets.
- The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
- The images of compact subsets can be chosen to cover X.
- Thus, these functions become functions on X which sums to a positive valued function.
- We divide by the sum.

An orbifold X is *orientable* if one can choose an atlas of charts where $\tilde{U}$ is given an orientation with G acting in an orientation-preserving manner and each imbedding of charts to another charts is orientation-preserving. For example, a reflection about a hypersurface is excluded and hence silvered boundary is excluded. (However, one can use densities to replace n-forms and can integrate.)

An n-form can be integrated on an orientable orbifold.

$$\int_{\tilde{U}} \omega = \frac{1}{|G|} \int_U \omega'$$

where $(\tilde{U}_i, G, \phi)$ is the chart for U. (Otherwise, one can define n-density to integrate.) Then any n-form can be integrated by using a partition of unity.

- Poincare duality pairing: For a compact orbifold X
$$\int : H^p(X) \otimes H_c^{n-q}(X) \to \mathbb{R}.$$
This is nondegenerate if X has a finite good cover.
- A cover of an orbifold is *good* if each U is of form $\mathbb{R}^n/G$ and all of its intersections is of the form. In this case, the standard differentiable form arguments work (See Bott-Tu). A compact orbifold has a finite good cover. (Note the confusing terminology here.)

4.5.1 *Bundles over orbifolds*

An *orbifold-bundle* (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that

- Let F be a smooth manifold with a Lie group $\mathbf{G}$ acting on it smoothly.
- Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
- Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), (U^{*,'}, G^{*,'}, \phi^{*,'})$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \rightarrow (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \rightarrow (U^{*,'}, G*.', \phi*.')$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
- We have

$$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$

for embeddings $(U, G, \phi) \rightarrow (U', G', \phi') \rightarrow (U'', G'', \phi'')$.
- If $F = \mathbf{G}$, then this is a principle orbifold bundle.

4.5.2 *Tangent bundles, Tensor bundles*

Given an orbifold, we can build a *tangent orbifold-bundle* by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p. We can build any tensor bundles in this way by letting $F = T^r_s(\mathbb{R}^n)$ and $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the induced map $T^r_s(\mathbb{R}^n) \rightarrow T^r_s(\mathbb{R}^n)$ of λ at p.

A *reduction* of Lie group to H means an injective homomorphism $H \rightarrow G$ which induces a bundle morphism of the principal bundle with Lie group H to the principal bundle with Lie group G.

An *affine frame bundle* is given by taking $F = A_n(\mathbb{R}^n)$ the space of affine frames and $\mathbf{G} = A(\mathbb{R})^n$, the Lie group of affine autormorphisms. An *affine tangent bundle* is given by taking $F = \mathbb{R}^n$ with the same Lie group.

A frame bundles is obtained by taking $F = F_n(\mathbb{R}^n)$ the space of frames in $\mathbb{R}^n$ and $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the induced map $F_n(\mathbb{R}^n) \rightarrow F_n(\mathbb{R}^n)$ of λ at p.

Orthogonal frame bundles can be built in this way. We let $F = O_n(\mathbb{R}^n)$ the space of orthonormal frames and let $\mathbf{G} = O(n, \mathbb{R})$ and $g_\lambda(p)$ be the induced map $O_n(\mathbb{R}^n) \rightarrow O_n(\mathbb{R}^n)$ of λ at p.

A *Riemannian metric* on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings or simply as a smooth section of symmetric covariant tensor bundle $ST^2(M)$ whose image lie in the positive definite forms. A Riemannian metric can be built using partition of unity again from any given Riemannian metrics on charts. Another way is to see this as a smooth map s from the orbifold O to the tensor bundle $T^2(O)$ so that each values lies in positive definite tensors so that $p \circ s : O \to O$ is the identity orbifold map.

Given a principle bundle P, one defines *connection* to be equivariant connections on each (U^*, G^*, ϕ^*) corresponding to (U, G, ϕ) of X and which are consistently defined under the embeddings. The *curvature* is also defined as $\mathcal{G}$-valued 2-form on O which comes from the curvature of each orbifold charts. Torsion is defined similarly.

A *linear connection* is a connection on a frame bundle or a tangent bundle with Lie group $GL(n, \mathbb{R})$. An *affine connection* is a connection on an affine frame bundle or an affine tangent bundle with Lie group $GL(n, \mathbb{R})$. Given an affine connection on an affine tangent bundle, a *geodesic* is defined as a smooth map from an open arc to O so that in each chart it lifts to a geodesic under the connection. As usual, a connection of a tangent bundle or a frame bundle is also considered an affine connection since we can always construct a canonical affine connection from a linear connection. The set of geodesics do not change here.

We can also replace the group with $O(n, \mathbb{R})$ by reduction of the group. This correspond to choosing a section to $T^2(O)$. Then the connections on the reduced tangent bundles are also called *affine connections*.

Finally, one can define an exponential map $Exp : TO \to O$: one defines the exponential map using the linear or affine connection and then patching up the consistent results.

Finally, using the groupoid language, we can define:

A *principle L-bundle* for a Lie group L over a Lie groupoid is a G-space P with a left action $L \times P \to P$ which maps $\pi : P \to G_0$ into a principle L-bundle and $(l.p).g = l.(p.g)$ for $p \in P, l \in L$ and $g : x \to y$.

One can see easily that this is an equivalent definition to above.

4.5.3 *Existence of locally finite good covering*

Prop 4.3. Let X be an orbifold. There exists a good covering: each open set is connected and charts have cells as cover and the intersection of any finite collection again has such properties.

Proof. Give X a Riemannian metric. Each point has an open neighborhood with an orthogonal action. Now choose sufficiently small ball centered at the origin so that it has a convexity property. (That is, any path can be homotoped into a geodesic.) Find a locally finite subcollection. Then intersection of any finite collection is still convex and hence has cells as cover. □

4.5.4 *Gauss-Bonnet theorem*

Assuming that X admits a finite smooth triangulation so that interior of each cell lies in singularity with locally constant isotopy groups, then we define the Euler characteristic to be

$$\chi(X) = \sum_k (-1)^{\dim s_k} 1/N_{s_k}$$

where s_k denotes the kth-cell and N_{s_k} the order of the isotropy group.

We note that such a triangulation always seem to exist always. (Proved in Verona [**?**)].)

Theorem 4.5.1. *(Allendoerfer-Weil, Hopf) Let M be a compact Riemannian orbifold of even dimension m. Then*

$$(2/O_m) \int_M K dw = \chi(M),$$

where O_m is the volume of the m-sphere.

The proof essentially follows that of Chern for manifolds.

4.6 Covering spaces of orbifolds

Let X be an orbifold. Let X' be an orbifold with a smooth map $p : X' \to X$ so that for each point x of X, there is a connected model (U, G, ϕ) and the inverse image of $p(\psi(U))$ is a union of open sets with models isomorphic to (U, G', π) where $\pi : U \to U/G'$ is a quotient map and G' is a subgroup of G. Then $p : X' \to X$ is a *covering* and X' is a *covering orbifold* of X.

Abstract monoid definition: If X' is a (X_1, X_0)-space and $p_0 : X'_0 \to X_0$ is a covering map, then X' is a *covering orbifold*.

We can see it as an orbifold bundle over X with discrete fibers. We can choose the fibers to be acted upon by a discrete group G (usually on the right), and hence a principal G-bundle. This gives us a regular (Galois) covering.

4.6.1 *Fiber product construction by Thurston*

Let us first review the fiber product constructions for ordinary covering space theory.

Let be Y a manifold. $\tilde{Y}$ a regular covering map $\tilde{p}$ with the automorphism group Γ. Let $\Gamma_i, i \in I$ be a sequence of subgroups of Γ.

- The projection $\tilde{p}_i : \tilde{Y} \times \Gamma_i\backslash\Gamma \to \tilde{Y}$ induces a covering $p_i : (\tilde{Y} \times \Gamma_i\backslash\Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y$ where Γ acts by
$$\gamma(\tilde{x}, \Gamma_i\gamma_i) = (\gamma(\tilde{x}), \Gamma_i\gamma_i\gamma^{-1})$$
- This is same as $\tilde{Y}/\Gamma_i \to Y$ since Γ acts transitively on both spaces.
- Fiber-products $\tilde{Y} \times \prod_{i\in I}\Gamma_i\backslash\Gamma \to \tilde{Y}$. Define left-action of Γ by
$$\gamma(\tilde{x}, (\Gamma_i\gamma_i)_{i\in I}) = (\gamma(\tilde{x}), (\Gamma_i\gamma_i\gamma^{-1})), \gamma \in \Gamma.$$
We obtain the fiber-product
$$(\tilde{Y} \times \prod_{i\in I}\Gamma_i\backslash\Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y.$$
Then this construction give us coverings with perhaps many components in the covering spaces. To understand this, suppose that Γ is a properly discontinuous and free action.

4.6.1.1 *Developable orbifold*

We can let Γ be a discrete group acting on a manifold $\tilde{Y}$ properly discontinuously but maybe not freely.

One can find a collection X_i of coverings so that

- $\Gamma_i = \{\gamma \in \Gamma | \gamma(X_i) = X_i\}$ is finite and if $\gamma(X_i) \cap X_i \neq \emptyset$, then γ is in Γ_i.
- The images of X_i cover $\tilde{Y}/\Gamma$.

Then $Y = \tilde{Y}/\Gamma$ has an *orbifold quotient* of $\tilde{Y}$ and Y is said to be *developable.*

In the above example, we can let Γ be a discrete group acting on a manifold $\tilde{Y}$ properly discontinuously but maybe not freely. Y^f is then the fiber product of orbifold maps $\tilde{Y}/\Gamma_i \to Y$.

4.6.1.2 *The doubling orbifolds*

A *mirror point* or *silvered point* is a singular point with the stablizer group $\mathbb{Z}_2$ acting as a reflection group. One can double an orbifold M with mirror points so that mirror-points disappear. (The double covering orbifold is orientable.)

- Let V_i be the neighborhoods of M with charts (U_i, G_i, ϕ_i).
- Define new charts $(U_i \times \{-1, 1\}, G_i, \phi_i^*)$ where G_i acts by $(g(x, l) = (g(x), s(g)l)$ where $s(g)$ is 1 if g is orientation-preserving and -1 if not and ϕ_i^* is the quotient map.

- For each embedding, $i : (W, H, \psi) \to (U_i, G_i, \phi_i)$ we define a lift $(W \times \{-1, 1\}, H, \psi^*) \to (U_i \times \{-1, 1\}, G_i, \phi_i^*$. This defines the gluing.
- The result is the doubled orbifold and the local group actions are orientation preserving.
- The double covers the original orbifold with Galois group $\mathbb{Z}_2$.

Prop 4.4. A doubled orbifold has no reflection with a hypersurface fixed set. Hence the set of regular points is dense open and connected.

Proof. Since there are no orientation reversing elements in the local group, the first statement is clear. If there are no reflections, then the singularity is of codimension two or greater and hence the set of regular points is dense open and path connected locally. Thus, the second statement follows. □

In the abstract groupoid definition, we simply let X_0' be the orientation double cover of X_0 where G-acts on X' preserving the orientation.

For example, if we double a corner-reflector, it becomes a cone-point.

- Clearly, manifolds are orbifolds. Manifold coverings provide examples.
- Let Y be a tear-drop orbifold with a cone-point of order n. Then this cannot be covered by any other type of an orbifold and hence is a universal cover of itself.
- A sphere Y with two cone-points of order p and q which are relatively prime.
- Choose a cyclic action of Y of order m fixing the cone-point. Then Y/Z_m is an orbifold with two cone-points of order pm and qm.

4.6.2 *Universal covering by fiber-product*

A *universal cover* of an orbifold Y is an orbifold $\tilde{Y}$ covering any covering orbifold of Y. We will now show that the universal covering orbifold exists by using fiber-product constructions. For this we need to discuss elementary neighborhoods. An *elementary* neighborhood is an open subset with a chart components of whose inverse image are open subsets with charts.

We can take the model open set in the chart to be simply connected. Then such an open set is elementary.

4.6.2.1 *Fiber-product for D^n/G_i*

If V is an orbifold D^n/G for a finite group G.

- Any covering is D^n/G_1 for a subgroup G_1 of G.
- Given two covering orbifolds D^n/G_1 and V/G_2, a covering morphism is induced by $g \in G$ so that $gG_1g^{-1} \subset G_2$.
- The covering morphism is in one-to-one correspondence with the double cosets of form G_2gG_1 for g such that $gG_1g^{-1} \subset G_2$.
- The covering automorphism group of D^n/G' is given by $N(G_1)/G_1$.

Given coverings $p_i : V/G_i \to V/G$ for $G_i \subset G$ for V homeomorphic to a cell, we form a fiber-product.

$$V^f = (V \times \prod_{i \in I} G_i \backslash G)/G \to V/G$$

If we choose all subgroups G_i of G, then any covering of V/G is covered by V^f induced by projection to G_i-factor (universal property)

4.6.2.2 *The construction of the fiber-product of a sequence of orbifolds*

Let $Y_i, i \in I$ be a collection of the orbifold-coverings of Y. We cover Y by elementary neighborhoods V_j for $j \in J$ forming a good cover. We take inverse images $p_i^{-1}(V_j)$ which is a disjoint union of V/G_k for some finite group G_k. Fix j and we form one fiber product by V/G_k by taking one from $p_i^{-1}(V_j)$ for each i. We form a fiber-product of $p_i^{-1}(V_j)$, which will essentially be the disjoint union of the above fiber products indiced by the product of the component indices for each i. Over regular points of V_j, this is the ordinary fiber-product. Now, we wish to patch these up using imbeddings. Let $U \to V_j \cap V_k$. We can assume $U = V_j \cap V_k$ which has a convex cell as a cover.

- We form the fiber products of $p_i^{-1}(U)$ as before which can be realized in V_j and V_k.
- Over the regular points in V_j and V_k, they are isomorphic. Then they are isomorphic.
- Thus, each component of the fiber-product can be identified.

By patching, we obtain a covering Y^f of Y with the covering map p^f. Note that Y^f is not necessarily connected.

4.6.2.3 *Thurston's example of fiber product*

Let I be the unit interval. Make two endpoints into silvered points. Then $I_1 = I$ is double covered by $\mathbf{S}^1$ with the deck transformation group $\mathbb{Z}_2$. Let p_1 denote the covering map. $I_2 = I$ is also covered by I by a map $x \mapsto 2x$ for $x \in [0, 1/2]$ and $x \mapsto 2 - 2x$ for $x \in [1/2, 1]$. Let p_2 denote this covering map. Then we determine the fiber product of p_1 and p_2: Cover I by $A_1 = [0, \epsilon), A_2 = (\epsilon/2, 1 - \epsilon/2), A_3 = (\epsilon, 1]$.

- Over A_1, I_1 has an open interval and I_2 has two half-open intervals. The fiber-product is a union of two copies of open intervals.
- Over A_2, the fiber product is a union of four copies of open intervals.
- Over A_3, the fiber product is a union of two copies of open intervals.
- By pasting considerations, we obtain a circle mapping 4-1 almost everywhere to I.

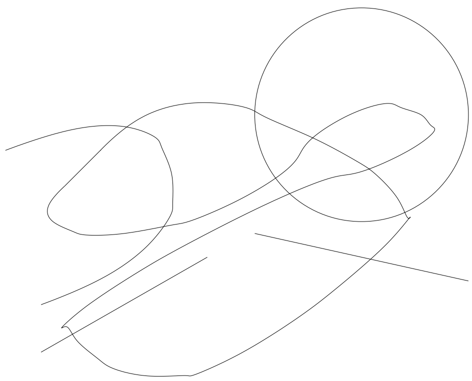


Fig. 4.1 ????

4.6.2.4 *The construction of the universal cover*

The collection of cover of an orbifold is countable up to covering isomorphisms preserving base points. (Cover by a countable good cover and for each elementary neighborhood, there is a countable choice.) We take each one with a different choices of base points. The base point is over a regular point y of Y and hence all are regular points. We call them (Y_i, y_i). We take a fiber product of (Y_i, y_i), $i = 1, 2, 3, ...$ and we take a connected component $\tilde{Y}$ containing the base point y^*. The base-point y^* also is regular.

Then for any cover (Y_i, y_i), there is a covering morphism $q_i : \tilde{Y} \to Y_i$ with $q_i(y_i) = y^*$ and so that $p_i \circ q_i = p$.

Prop 4.5. A universal cover has a open dense and connected set of regular points.

Proof. A universal cover has a morphism to a double of the orbifold. Any point mapping to a regular point is also regular. The set of such points is also dense and open and locally path connected. Hence the proposition follows. $\square$

Theorem 4.6.1. *The universal cover is unique up to covering orbifold-isomorphisms by the universality property.*

Proof. If (Y', y') is another universal cover. Then it arises in the list of covers and hence there is a covering morphism $q : \tilde{Y} \to Y'$ with $q(y') = y^*$. Conversely, we have a morphism $p' : \tilde{Y} \to Y'$ with $p'(y^*) = y^*$. We obtain a morphism $p' \circ q : \tilde{Y} \to \tilde{Y}$ fixing y^*. By restricting it to regular subset, we find that it restricts to identity in the regular part of $\tilde{Y}$. Since the regular part is open and dense, $p' \circ q$ is the identity. Similarly, so is $q \circ p'$. $\square$

4.6.2.5 *Properties of the universal cover*

The group of automorphisms of $\tilde{Y}$ is called the *fundamental group* and is denoted by $\pi_1(Y)$.

Prop 4.6.

- $\pi_1(Y)$ acts transitively on $\tilde{Y}$ on fibers of $\tilde{p}^{-1}(x)$ for each x in Y.
- $\tilde{Y}/\pi_1(Y) = Y$.
- Any covering of Y is of form $\tilde{Y}/\Gamma$ for a subgroup Γ of $\pi_1(Y)$.
- The isomorphism classes of coverings of Y is the set of conjugacy classes of subgroups of $\pi_1(Y)$.

Proof. Let y be a base-point of Y. We change the base point of $\tilde{Y}$ to any point of $\tilde{p}^{-1}(y)$. Then there are always a morphism $q : (\tilde{Y}, y^*) \to (\tilde{Y}, z)$. We find an inverse to q by finding $t = q^{-1}(y^*)$. Then there exists a morphism $q' : (\tilde{Y}, y^*) \to (\tilde{Y}, t)$. Hence, $q \circ q'(y^*) = y^*$. Thus, q' is the inverse and q is an automorphism. Thus, $\pi_1(Y)$ acts transitively on $\tilde{p}^{-1}(y)$.

Given a point x, we find a path γ in Y with endpoints x and y. Then γ lifts to a smooth curve in $\tilde{Y}$ with endpoints a point of $\tilde{p}^{-1}(x)$ and $\tilde{p}^{-1}(y^*)$. We see that $\pi_1(Y)$ also acts transitively on the set of lifts. Since we can find a lift starting from any point of $\tilde{p}^{-1}(x)$, we see that $\pi_1(Y)$ acts transitively on $\tilde{p}^{-1}(y)$.

We see that $\tilde{Y}/\pi_1(Y)$ is clearly in one-to-one correspondence with Y. The charts are also compatible.

For a covering $Y' \to Y$, there is a covering morphism $p' : \tilde{Y} \to Y'$. Therefore, Y' is a quotient orbifold of $\tilde{Y}$.

Given two coverings $Y_1 \to Y$ and $Y_2 \to Y$, an isomorphism $f : Y_1 \to Y_2$ lifts to a diffeomorphism $\tilde{Y} \to \tilde{Y}$. We choose a morphism fixing y^* by multiplying by an element of $\pi_1(Y)$. By restricting to the regular part, we see that the morphism is the identity map and f is induced by an element of $\pi_1(Y)$. Since $Y_1 = \tilde{Y}/\Gamma_1$ and $Y_2 = \tilde{Y}/\Gamma_2$, it follows that Γ_1 and Γ_2 are conjugate. The converse is also simple. $\square$

We see that given a covering $\tilde{Y}/\Gamma$, the group of automorphism is $N(\Gamma)/\Gamma$. A covering is regular if and only if Γ is normal.

A *good orbifold* is an orbifold with a cover that is a manifold. A *very good orbifold* is an orbifold with a finite cover that is a manifold. A good orbifold has a simply-connected manifold as a universal covering space since it has a covering space that is a manifold and the universal covering orbifold must cover this manifold and hence the universal covering space has to be a manifold.

4.6.2.6 *Induced homomorphism of the fundamental group*

Given two orbifolds Y_1 and Y_2 and an orbifold-diffeomorphism $g : Y_1 \to Y_2$. Then the lift to the universal covers $\tilde{Y}_1$ and $\tilde{Y}_2$ is also an orbifold-diffeomorphism. Furthermore, once the lift value is determined at a point, then the lift is unique.

An *isotopy* $F : Y_1 \times I \to Y_2$ is an orbifold-map such that for each $t \in I$, F restricts to a diffeomorphism of $Y_1 \times \{t\}$ to Y_2.

Prop 4.7. An isotopy $f_t : Y_1 \to Y_2$ of orbifold-maps lift to an isotopy in the universal covering orbifold $\tilde{f}_t : \tilde{Y}_1 \to \tilde{Y}_2$ for each $t \in I$ unique up to a choice of $\tilde{f}_0(y)$.

Proof. We consider regular parts and model neighborhoods where the lift clearly exists uniquely for each t. The map $t \mapsto f_t(y)$ for a regular base point y of Y is a path in Y. Then $f_t(y)$ is regular for all $t \in I$. This lifts to a smooth path $\tilde{\gamma} : t \mapsto p^{-1}(f_t(y))$. By post-composing with elements of $\pi_1(Y)$ if necessary, we can make sure that $\tilde{f}_t(y) = \tilde{\gamma}(t)$ for each t. Now, we can verify that $\tilde{f}_t$ form an isotopy. $\square$

Given orbifold-diffeomorphism $f : Y \to Z$ which lift to a diffeomorphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$, we obtain $f_* : \pi_1(Y) \to \pi_1(Z)$. If g is homotopic to f, then $g_* = f_*$.

4.7 The path-approach to the universal covering spaces

4.7.1 *G-paths*

The notion of G-paths generalize the notion of paths to those on groupoids: Given an etale groupoid X, a *G-path* $c = (g_0, c_1, g_1, ..., c_k, g_k)$ over a subdivision $a = t_0 \leq t_1 \leq ... \leq t_k = b$ of interval $[a, b]$ consists of

- continuous maps $c_i : [t_{i-1}, t_i] \to X_0$
- elements $g_i \in X_1$ so that $s(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, .., k-1$ and $t(g_i) = c_i(t_i)$ for $i = 1, .., k$.

The initial point is $t(g_0)$ and the terminal point is $s(g_k)$.

The two operations define an equivalence relation:

- Subdivision: Add new division point t'_i in $[t_i, t_{i+1}]$ and $g'_i = 1_{c_i(t'_i)}$ and replacing c_i with c'_i, g'_i, c''_i where c'_i, c''_i are restrictions to $[t_i, t'_i]$ and $[t'_i, t_{i+1}]$.
- Replacement: replace c with $c' = (g'_0, c'_1, g'_1, .., c'_k, g'_k)$ as follows. For each i choose continuous map $h_i : [t_{i-1}, t_i] \to X_1$ so that $s(h_i(t)) = c_i(t)$ and define $c'_i(t) = t(h_i(t))$ and $g'_i = h_i(t_i) g_i h^{-1}_{i+1}(t_i)$ for $i = 1, .., k-1$ and $g'_0 = g_0 h^{-1}_1(t_0)$ and $g'_k = h_k(t_k) g_k$.

All paths are defined on $[0, 1]$ from now on. Given two paths $c = (g_0, c_1, .., c_k, g_k)$ over $0 = t_0 \leq t_1 \leq ... \leq t_k = 1$ and $c' = (g'_0, c'_1, .., c'_{k'}, g'_{k'})$ such that the terminal point of c equals the initial point of c', the *composition* $c * c'$ is the G-path $c'' = (g''_0, c''_1, .., g''_{k+k'})$ so that

- $t''_i = t_i/2$ for $i = 0, .., k$ and $t''_i = 1/2 + t'_{i-k}/2$ and
- $c''_i(t) = c_i(2t)$ for $i = 1, .., k$ and $c''_i(t) = c'_{i-k}(2t-1)$ for $i = k+1, ..., k+k'$.
- $g''_i = g_i$ for$i = 1, .., k-1$ and $g''_k = g_k g'_0, g''_i = g'_{i-k}$ for $i = k + 1, .., k + k'$.

The inverse c^{-1} is $(g'_0, c'_1, ..., c'_k, g'_k)$ over the subdivision where $t'_i = 1 - t_i$ so that $g'_i = g^{-1}_{k-i}$ and $c'_i(t) = c_{k-i+1}(1-t)$.

4.7.1.1 *Homotopies of G-paths*

There are two types of homotopies

- equivalences
- An *elementary homotopy* is a family of G-paths $c^s = (g_0^s, c_1^s, ..., g_k^s)$ over the subdivision $0 = t_0^s \leq t_1^s \leq ... \leq t_k^s = 1$ so that t_k^s, g_i^s, c_i^s depends continously on s.

Two G-paths a and b are *homotopic* if there is a sequence of G-paths $a = a_1, a_2, ..., a_n = b$ so that a_i and a_{i+1} are either equivalent or there is an elementary homotopy between them.

A homotopy class of c is denoted $[c]$. $[c * c']$ is well-defined in the homotopy classes $[c]$ and $[c']$. Hence, we define $[c] * [c'] = [c * c']$.

We have associativity $[c * (c' * c'')] = [(c * c') * c'']$.

The constant path e_x at x is given as $(1_x, x, 1_x)$. Then $[c * c^{-1}] = [e_x]$ if the initial point of c is x and $[c^{-1} * c] = [e_y]$ if the terminal point of c is y. Thus, $[c]^{-1} = [c^{-1}]$.

4.7.1.2 *Fundamental group $\pi_1(X, x_0)$*

A *loop* is a G-path with the identical initial and terminal points. The *fundamental group* $\pi_1(X, x_0)$ based at $x_0 \in X_0$ is the group of homotopy classes of loops based at x_0. The associativity, identity and inverse properties are proven above.

A continuous homomorphism $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

This is well-defined up to conjuations. An orbifold-equivalence induces an isomorphism.

Theorem 4.7.1. *(Seifert-Van Kampen theorem) Let X be an orifold. Let $X_0 = U \cup V$ where U and V are open and $U \cap V = W$. Assume that the groupoid restrictions G_U, G_V, G_W to U, V, W are connected. And let $x_0 \in W$. Then $\pi_1(X, x_0)$ is the quotient group of the free product $\pi_1(G_U, x_0) * \pi_1(G_V, x_0)$ by the normal subgroup generated by $j_U(\gamma) j_W(\gamma^{-1})$ for $\gamma \in \pi_1(G_W, x_0)$ for j_U the induced homomorphism $\pi_1(G_W, x_0) \to \pi_1(G_U, x_0)$ and j_V the induced homomorphism $\pi_1(G_W, x_0) \to \pi_1(G_V, x_0)$.*

The proof is omitted but is essentially same as the elementary topology proof.

4.7.1.3 *Examples*

- Let a discrete group Γ act on a connected manifold X_0 properly discontinuously. Then (Γ, X_0) has an orbifold structure. Any loop can be made into a G-path $(1_x, c, \gamma)$ so that $\gamma(x) = c(1)$. and $c(0) = x$. Thus, there is an exact sequence

$$1 \to \pi_1(X_0, x_0) \to \pi_1((\Gamma, X_0), x_0) \to \Gamma \to 1$$

- A two-orbifold that is a disk with an arc silvered has the fundamental group isomorphic to Z_2.
- A two-dimensional orbifold with cone-points which is boundariless and with no silvered point.
- A tear drop: A sphere with one cone-point of order n has the trivial fundamental group
- An annulus with one boundary component silvered has a fundamental group isomorphic to $Z \times Z_2$. This can be seen since our orbifold is covered by an annulus by an action of Z_2 which fixes the middle circle of the annulus.

The fundamental group can be computed by removing open-ball neighborhoods of the cone-points and using Van-Kampen theorem: Suppose that a two-dimensional orbifold has boundary and silvered points. Then remove open-ball neighborhoods of the cone-points and corner-reflector points. Then the fundamental group of remaining part can be computed by Van-Kampen theorem by taking open neighborhoods of silvered boundary arcs. Finally, adding the open-ball neighborhoods, we compute the fundamental group.

The fundamental group of a three-dimensional orbifold can be computed similarly.

4.7.1.4 *Seifert fibered* 3*-manifold Examples*

We can obtain a 2-orbifold from a Seifert fibered 3-manifold M: let X_0 be the union of open disks transversal to the fibers, and let X_1 will be the arrows obtained by the flow.

The orbifold X will be a 2-dimensional orbifold with cone-points whose orders are obtained as the numerators of the fiber-order.

The fundamental group of X is then the quotient of the ordinary fundamental group $\pi_1(M)$ by the central cyclic group $\mathbb{Z}$ generated by the generic fiber.

4.7.2 *Covering spaces and the fundamental group*

One can build the theory of covering spaces using the fundamental group. We first review the relationship of the homotopy group of G-paths to covering spaces first.

Let us be given a covering $X' \to X$. For every G-path c in X, there is a lift G-path in X'. If we assign the initial point, the lift is unique. If c' is homotopic to c, then the lift of c' is also homotopic to the lift of c provided the initial points are the same. From this it follows that the induced homomorphism $\pi_1(X', x_0') \to \pi_1(X, x_0)$ is injective.

A map from a simply connected orbifold to an orbifold lifts to a cover. The lift is unique if the base-point lift is assigned. Thus, a simply connected cover of an orbifold covers any cover of the given orbifold. From this, we can show that the fiber-product construction is simply-connected. (Hence, the fiber-product constructed cover is a universal cover in the sense given here.)

Two simply-connected coverings of an orbifold are isomorphic and if base-points are given, we can find an isomorphism preserving the base-points.

Theorem 4.7.2. *A simply-connected covering of an orbifold X is a Galois-covering with the Galois-group isomorphic to $\pi_1(X, x_0)$.*

Proof. Consider $p^{-1}(x_0)$. Choose a base-point $\tilde{x}_0$ in it. Given a point of $p^{-1}(x_0)$, connected it with $\tilde{x}_0$ by a path. The paths map to the fundamental group. The Galois-group acts transitively on $p^{-1}(x)$. Hence the Galois-group is isomorphic to the fundamental group. □

4.7.2.1 *The existence of the universal cover using path-approach*

The construction follows that of the ordinary covering space theory.

- Let $\hat{X}$ be the set of homotopy classes $[c]$ of G-paths in X with a fixed starting point x_0.
- We define a topology on $\hat{X}$ by open set $U_{[c]}$ that is the set of paths ending at a simply-connected open subset U of X with homotopy class $c * d$ for a path d in U.
- Define a map $\hat{X} \to X$ sending $[c]$ to its endpoint other than x_0.
- Define a map $\hat{X} \times X_1 \to \hat{X}$ given by $([c], g) \to [c * g]$. This defines a right G-action on $\hat{X}$. This makes $\hat{X}$ into a bundle.
- Define a left action of $\pi_1(X, x_0)$ on $\hat{X}$ given by $[c] * [c'] = [c * c']$ for

$[c'] \in \pi_1(X, x_0)$. This is transitive on fibers.
- We show that $\hat{X}$ is a simply connected orbifold.

4.8 Helpful references

For compact group actions, see [**(author?)** (Bredon)], [**(author?)** (Hsiang)]. Good references for triangulation is [**(author?)** (Illman)] and [Illman (6)] under group actions. For triangulations of stratified spaces, and hence orbifolds, see [**?**)] and [**(author?)** (Weinberger)]. The work [**?**)] seems to be most self-contained.

For general introduction to the orbifold theory see [Thurston (10)] and [**(author?)** (Matsumoto and Montesinos-Amilibia)]. Also, the original papers [Satake (8)] and [Satake (9)] are also very readable. The book by Adem et al [Adem, Leida, and Ruan (1)] and [**(author?)** (Bridson and Haefliger)] treat orbifolds as groupoids. Read [Moerdijk (7)] and [**(author?)** (Moerdijk and D.A. Plonk)] for this approach in detail. [**(author?)** (Haefliger)] and the papers and Chapter 13 of [**(author?)** (Ratcliffe)] treats the path approaches to the covering spaces. See also [**?**)]. Thurston's chapter [Thurston (10)] and the paper [**(author?)** (Choi)] give a covering space theory as fiber products.

Chapter 5

Topology of 2-orbifolds

2-orbifold topological constructions

5.1 2-orbifolds

We now wish to concentrate on 2-orbifolds to illustrate more concretely. In many cases, the theory is much easier to understand.

To study singularities, we simply have to classify finite groups in $O(2)$ since we are looking at finite subgroups of $GL(2, \mathbb{R})$: These are: $\mathbb{Z}_2$ acting as a reflection group or a rotation group of angle $\pi/2$, a cyclic groups C_n of order ≥ 3 and dihedral groups D_n of order ≥ 4. Recall that 2-orbifold have three types of singularities: silvered points in open arcs, isolated cone-points, and isolated corner-reflector points. The singular points of a two-dimensional orbifold fall into three types:

(i) The mirror point: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflections on the y-axis.
(ii) The cone-points of order n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.
(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n.

The singular strata associated with conjugate local groups are as follows: the silvered point is a subset of arc of silvered points which may have an end point in the boundary of the orbifold. The other types have isolated points as strata. Note that the silvered arc may end in a corner-reflector of order ≥ 2 also but not at a cone-point by the local group considerations.

- On the boundary of a surface with a corner, one can take mutu-

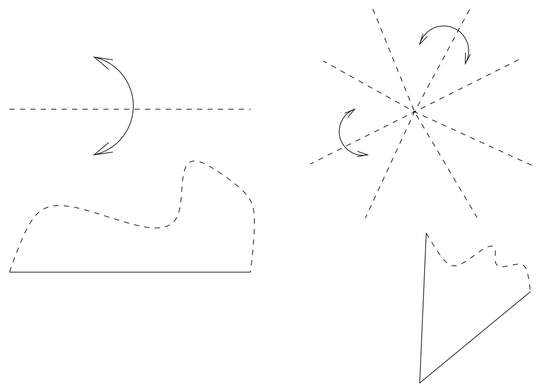


Fig. 5.1 The actions here are isometries on $\mathbb{R}^2$.

ally disjoint open arcs ending at corners. If two arcs meet at a corner-point, then the corner-point is a *distinguished one*. If not, the corner-point is *ordinary*. The choice of arcs will be called the *boundary pattern*.

- As noted above, given a surface with corner and a collection of discrete points in its interior and the boundary pattern, it is possible to put an orbifold structure on it so that the interior points become cone-points and the distinguished corner-points the corner-reflectors and boundary points in the arcs the silvered points of any given orders.

I need Hirsch's book

Theorem 5.1.1. *Theorem: Any* 2*-orbifold is obtained from a smooth surface with corner by silvering some arcs and putting cone-points and corner-reflectors. A* 2*-orbifold is classified by the underlying smooth topology of the surface with corner and the number and orders of cone-points, corner-reflectors, and the boundary pattern of silvered arcs.*

Proof. basically, strata-preserving isotopies. □

5.1.1 *The triangulations of* 2*-orbifolds and classification*

For 2-orbifolds, the Riemannian metric and triangulation can be approached in more simple manner.

Prop 5.1. One can put a Riemannian metric on a 2-orbifold so that the boundary is a union of geodesic arcs and each corner-reflector have angles

π/n for its order n and the cone-points have angles $2\pi/n$. One can give a triangulation by smooth triangles so that slivered arcs and boundary curves in the union of 1-skeletons and corner-reflectors and cone-points are in 0-skeletons.

Proof. First construct such a metric on the boundary by putting such metrics on the boundary by using a broken geodesic in the euclidean plane and around the cone points and then using partition of unity.

By removing open balls around cone-points and corner-reflectors, we obtain a smooth surface with corners.

Find a smooth triangulation of so that the interior of each side is either completely inside the boundary with the corners removed. Finally we extend the triangulations by cone-construction to the interiors of the removed balls. □

There is a general proof... Cornered manifold, markings....State it...

5.2 Topological operations on 2-orbifolds: constructions and decompositions

We will now study the question of how to construct and decompose 2-orbifolds:

- Classifications of 1-dimensional suborbifolds of 2-orbifolds
- Definition of splitting and sewing of 2-orbifolds
- Regular neighborhoods of 1-orbifolds
- Reinterpretation of splitting and sewing.
- Identification interpretations of splitting and sewing

5.2.1 *Classifications of 1-dimensional suborbifolds of 2-orbifolds*

A *suborbifold* Q' on a subspace $X_{Q'} \subset X_Q$ is the subspace so that each point of $X_{Q'}$ has a neighborhood in X_Q modeled on an open subset U of $\mathbb{R}^n$ with a finite group Γ preserving $U \cap \mathbb{R}^d$ where $\mathbb{R}^d \subset \mathbb{R}^n$ is a proper subspace, so that $(U \cap \mathbb{R}^d, \Gamma')$ is in the orbifold structure of Q'. Here Γ' denotes the restricted group of Γ to $U \cap \mathbb{R}^d$, which is in general a quotient group.

Note here that for a finite group G, there are always a complementary G-invariant subspace for a G-invariant subspace and a point on it given a smooth action of G, which is linear locally. Thus, Γ' can lift to an injection to Γ onto a subgroup with the set of fixed points the complement of $U \cap \mathbb{R}^n$. Thus Γ' is a direct summand of Γ.

This agrees with the orbifold book... Check the general case...

5.2.1.1 *Classifications of* 1*-dimensional suborbifolds of* 2*-orbifolds*

A compact 1-orbifold is either a closed arc, a segment, a segment with one silvered endpoint, or a segment with two silvered end-point.

A *nicely* imbedded suborbifold is an imbedded suborbifold so that its boundary is in the boundary of the ambient orbifold so that each point of the boundary has a neighborhood modelled on a half space H^n with another half space H^m imbedded in it. A properly and nicely imbedded 1-orbifold in a 2-orbifold with boundary is either avoids the singular sets in its topological interior or is entirely contained in a singular set. In the former case we have:

- No silvered-point case: An imbedded closed arc avoiding boundary or singular points or a segment with two endpoints in the boundary avoiding singularities.
- One silvered-point case: A segment with silvered endpoint at a cone-point of order two or in the interior of a silvered arc and the other endpoint in the boundary.
- Two silvered-point case: A segment with silvered endpoints at cone-points or order two or in the interiors of silvered arcs.

If the 1-orbifold is in the singular set, we classify them as below:

- No silvered-point case: a segment in the interior of a silvered edge.
- One silvered-point case: a segment in a silvered edge with one endpoint in a corner reflector of order two and the other in the interior of the silvered edge.
- Two silvered-point case: a segment in a silvered edge with two endpoints in a corner-reflector of order two.

5.2.1.2 *Orbifold Euler-characteristic for* 2*-orbifolds due to Satake*

We defined the Euler characteristic to be

$$\chi(X) = \sum_{c_i} (-1)^{\dim(c_i)} (1/|\Gamma(c_i)|),$$

where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group Γ_i associated with c_i.

If X is finitely covered by another orbifold X', then $\chi(X') = r\chi(X)$ where r is the number of sheets for regular points. This follows since the sum of the order of local groups in the inverse image of the elementary neighborhood is always r.

The Euler-characteristic of 1-orbifold is as follows: a circle O, a segment 1, a segment with one silvered-point 1/2, a full 1-orbifold O.

For 2-orbifolds Σ_1, Σ_2 meeting in a compact 1-orbifold Y in the interior forming a 2-orbifold Σ as a union, we have the following additivity formula:

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - \chi(Y), \tag{5.1}$$

To be verified by counting cells with weights since the orders of singular points in the boundary orbifold equal the ambient orders.

Suppose that a 2-orbifold Σ with or without boundary has the underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j and n_Σ boundary full 1-orbifolds. Then the following generalized *Riemann-Hurwitz formula* is very useful:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_{j=1}^{n}\left(1 - \frac{1}{r_j}\right) - \frac{1}{2}n_\Sigma, \tag{5.2}$$

which is proved by a doubling argument and cutting and pasting.

5.2.2 *Geometrization of 2-orbifold: partial result*

Prop 5.2. Let S be a 2-orbifold whose underlying space is a disk or a 2-sphere and has more than two cone-points of orders $p_1, p_2, p_3, \ldots$. Then S is very good and so is regularly covered by a compact surface.

Proof. First, cover a double cover. Only cone-points...

n this case, O is easily shown to be a nontrivial quotient orbifold of a 2-sphere, or a euclidean space $\mathbb{R}^2$, or a hyperbolic space by a discrete subgroup of isometry group.

Use reflection groups....

Selberg's lemma

See for example Beardon... □

Nice pictures This might be more good at earlier chapters....

5.2.3 *Good and very good and bad 2-orbifolds*

The purpose of this section is to prove Theorem 5.2.1.

It is fairly easy to distinguish between the good and bad 2-orbifolds as Thurston shows [**?**]].

Since we know the existence of the universal cover of orbifolds from Chapter 4, we can cover any 2-orbifold S with a simply-connected 2-orbifold $\tilde{S}$. If these are silvered points, then we can double $\tilde{S}$ and obtain a double cover. Thus, $\tilde{S}$ has no silvered points.

Suppose that $\tilde{S}$ is not compact and has no singular points. Then $\tilde{S}$ is a manifold and S is good.

Suppose that $\tilde{S}$ is not compact and has some singular points. This is not possible. Suppose that there are cone points. We can remove a disk-neighborhood D of the cone-point of order say p for an integer $p > 1$. Then we cover $\tilde{S} - D$ by a p-fold cyclic cover by taking a completely imbedded arc from p avoiding singularities. Hence, by cutting and pasting p-copies of these we again obtain a nontrivial covering orbifold.

Suppose that $\tilde{S}$ is compact. Then the base space of $\tilde{S}$ is simply connected since otherwise we have a covering. Thus $\tilde{S}$ is a 2-sphere or a 2-disk.

Suppose first that $\tilde{S}$ is a 2-disk. If there are more than one cone-point, then by using a separating arc avoiding singular points, we again obtain nontrivial coverings. If $\tilde{S}$ has a unique singularity of order p, then $\tilde{S}$ can be covered by a disk in a p-fold way.

Suppose that $\tilde{S}$ is a 2-sphere and has more than two cone-points of orders $p_1, p_2, ...$ Then $\tilde{S}$ double-covers a disk O with mirrored edges and corner reflectors of corresponding orders $2p_1, 2p_2, ...$ where there are at least three or more. By Proposition 5.2, $\tilde{S}$ is good.

Finally suppose that $\tilde{S}$ is a 2-sphere with two cone-points of order m and n. If m and n have a common divisor p, then $\tilde{S}$ has a p-fold cover by a sphere with two cone points of order m/p and n/p.

A sphere with cone points of order p and q with p and q relatively prime is not covered by a manifold since we can show that the fundamental group is trivial by Van Kampen theorem. (See ???) A sphere with one cone point is also not covered by a manifold by the same reason.

Hence, we showed that except for a sphere with one or two singular points with orders m and n where $m \neq n$ is a bad orbifold. So is a disk with two edges silvered and two corner-reflectors of order m and n where $m \neq n$ are bad.

Theorem 5.2.1. *A sphere with one or two singular points with orders m and n where $m \neq n$ is a bad orbifold. So is a disk with two edges silvered and two corner-reflectors of order m and n where $m \neq n$ are bad. Except for these two, every other orbifold is good. Furthermore, they are very good.*

Proof. We need to show the final statement only. As above the orbifolds have cone-points only as singular points.

If the underlying space is of euler characteristic ≥ 1, then there is a covering by an orbifold whose underlying spaces are spheres or disks. This was studied above and was shown to be bad or is good. The good ones are very good according to Proposition 5.2.

Now suppose that the euler characteristic of the underlying space is ≤ 0. There exists a disk D containing all the cone-points. Since D admits a Euclidean or hyperbolic structure, it follows that there is a a finite covering orbifold that is very good. The boundary component of D is covered by m boundary components of the covering surface S and each component of S covers the boundary component of D by n-fold covering by Proposition 5.2. The closure of the complement is a surface S of negative euler characteristic and hence has infinite first homology. We map the homology to $\mathbb{Z}/n$ sending the boundary component class to 1. Then the kernel of the map gives us a finite covering S'' of the complement S'. We see that S'' has one boundary component mapping to S' in a n-fold way. Hence by attaching copies of S'' for each boundary component of S, we obtain a very good cover of the original orbifold. □

I am proving here $\chi < 0$ imply very good. How to do this? Probably this after the proof of hyperbolizations.

5.2.4 *Definition of Splitting and sewing 2-orbifolds*

We will assume that the orbifolds here are very good.

Let S be a very good orbifold so that its underlying space X_S is a pre-compact open surface with a path-metric admitting a compactification to a surface with boundary. Such a metric always exists and the topology of the compactification is unique up to homeomorphism type. Let $\hat{S}$ be a very good cover, that is, a finite regular cover, of S, so that S is orbifold-diffeomorphic to $\hat{S}/F$ where F is a finite group acting on $\hat{S}$.

Since $X_{\hat{S}} = \hat{S}$ is also pre-compact and has a path-metric, complete it to obtain a compact surface $X'_{\hat{S}}$. The action of the group F extends to $\hat{S}$ by the path-metric. Then $X'_{\hat{S}}/F$ with the quotient orbifold structure is said to be the *orbifold-completion* of S.

- Let S be a 2-orbifold with an embedded circle or a full 1-orbifold l in the interior of S. The completion S' of $S - l$ is said to be obtained from *splitting* S along l. Since $S - l$ has an embedded copy in S', we see that there exists a map $S' \to S$ sending the copy to $S - l$. Let l' denote the boundary component of S corresponding to l under the map.

- Conversely, S is said to be obtained from *sewing* S' *along* l'.
- If the interior of the underlying space of l lies in the interior of the underlying space of S, then the components of S' are said to be *decomposed components of* S *along* l, and we also say that S *decomposes* into S' along l.
- Of course, if l is a union of disjoint embedded circles or full 1-orbifolds, the same definition holds.

A boundary point has a neighborhood which is realized as a quotient of an open ball by a $\mathbb{Z}_2$-action generated by a reflection about a line.

There are two distinguished classes of splitting and sewing operations:

A simple closed curve boundary component can be made into a set of mirror points and conversely in a unique manner.

A boundary full 1-orbifold can be made into a 1-orbifold of mirror points and two corner-reflectors of order two and conversely in a unique manner: (a boundary point has a neighborhood which is a quotient space of a dihedral group of order four acting on the open ball generated by two reflections.) The forward process is called *silvering* and the reverse process *clarifying*.

5.2.5 ***Regular neighborhoods of*** **1*-orbifold***

5.2.5.1 *The classification of Euler-characteristic zero orbifold*

Let A be a compact annulus with boundary. The quotient orbifold of an annulus has Euler characteristic zero.

Prop 5.3. From Riemann-Hurwitz equation, all of the Euler characteristic zero 2-orbifolds with nonempty boundary is as follows:

(1) an annulus, (2) a Möbius band, (3) an annulus with one boundary component silvered (a *silvered annulus*),
(4) a disk with two cone-points of order two with no mirror points (a $(;2,2)$-*disk*),
(5) a disk with two boundary 1-orbifolds, two edges (*a silvered strip*),
(6) a disk with one cone-point and one boundary full 1-orbifold (a *bigon with a cone-point of order two*), that is, it has only one edge, and
(7) a disk with two corner-reflectors of order two and one boundary full 1-orbifold (a *half-square*). (It has three edges.)

Proof. To prove this, notice that the underlying space must have a non-negative Euler characteristic and Riemann-Hurwitz formula. When the

Euler characteristic of the space is zero, there are no cone-points, corner-reflectors, (1)(2)(3).

Suppose now that the underlying space is a disk. If there are no singular points in the boundary, then (4) holds as there has to be exactly two cone-points of order two. If two boundary full 1-orbifolds, then no singular points in the interior and no corner-reflector can exist; thus, (5) holds.

Suppose that exactly one boundary full 1-orbifold exist. If a cone-point exists, then it has to be a unique one and of order two. (6) holds. If there are no cone-points, but corner-reflectors, then exactly two corner-reflectors of order two and no more. (7) □

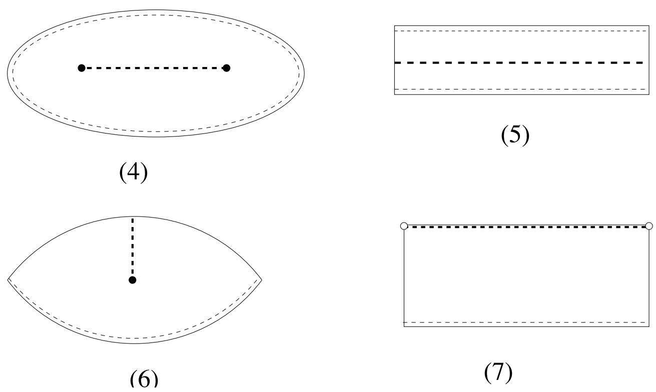


Fig. 5.2 ???.

5.2.5.2 *Regular neighborhoods of* 1*-orbifold*

A circle or a 1-orbifold l in the interior of a 2-orbifold S is not homotopic to a point. as we can see from the universal cover of S. l has a neighborhood of zero Euler characteristic considering its very good cover. Since the inverse image of l consists of closed curves which represent generators, we deduce that l is contained in the neighborhood as follows.

- For (1) and (2), l is the closed curve representing the generator of the fundamental group;
- For (3), l is the mirror set that is a boundary component;
- For (4), l is the arc connecting the two cone-points unique up to homotopy;

- For (5), l is an arc connecting two interior points of two edges respectively;
- For (6), l is an arc connecting an interior point of an edge and the cone-point of order two;
- For (7), the edge in the topological boundary connecting the two corner-reflectors of order two.

Given a 1-orbifold l and a neighborhood N of it in some ambient 2-orbifold, N is said to be a *regular neighborhood* if the pair (N, l) is diffeomorphic to one of the above.

Prop 5.4. A 1-orbifold in a good 2-orbifold has a regular neighborhood which is unique up to isotopy.

Proof. The existence is proved above. The uniqueness up to isotopy is proved as follows: Each regular neighborhood fibers over a 1-orbifold with fibers connected 1-orbifolds in the orbifold sense. A regular neighborhood can be isotoped into any other regular neighborhood by contracting in the fiber directions. To see this, we can modify the proof of Theorem 5.3 in Chapter 4 of Hirsch to be adopted to an annulus with a finite group acting on it and an imbedded circle. □

5.2.6 *Splitting and sewing on 2-orbifolds reinterpreted*

Let l be a 1-orbifold embedded in the interior of an orbifold S. If one removes l from the interior of a regular neighborhood, we obtain either a union of one or two open annuli, or a union of one or two open silvered strip. In (2)-(4), an open annulus results. For (1), a union of two open annuli results. For (6)-(7), an open silvered strip results. For (5), we obtain a union of two open silvered strips. These can be easily completed to be a union of one or two compact annuli or a union of one or two silvered strips respectively.

We can complete $S - l$ in this manner: We take a closed regular neighborhood N of l in S. We remove $N - l$ to obtain the above types and complete it and re-identify with $S - l$ to obtain a compactified orbifold. This process is the splitting of S along l.

Conversely, we can describe sewing: Take an open annular 2-orbifold N which is a regular neighborhood of a 1-orbifold l:

- Suppose that l is a circle. We obtain $U = N - l$ which is a union of one or two annuli.

- Take an orbifold S' with a union l' of one (resp. two) boundary components which are circles.
- Take an open regular neighborhood of l' and remove l' to obtain V.
- U and V are the same orbifold. We identify $S' - l'$ and $N - l$ along U and V.
- This gives us an orbifold S, and it is easy to see that S is obtained from S' by sewing along l'.
- l corresponds to a 1-orbifold l'' in S in a one-to-one manner. We can obtain (1),(2),(3)-type neighborhoods of l'' in this way. The operation in case (1) is said to be *pasting*, in case (2) *cross-capping*, and in case (3) *silvering* along simple closed curves.

- Suppose that l is a full 1-orbifold. $U = N - l$ is either an open annulus or a union of one (resp. two) silvered strips.
- The former happens if N is of type (4) and the latter if N is of type (5)-(7).
- In case (4), take an orbifold S' with a boundary component l' a circle. Then we can identify U with a regular neighborhood of l' removed with l' to obtain an orbifold S. Then l corresponds a full 1-orbifold l'' in S in a one-to-one manner. l'' has a type-(4) regular neighborhood. The operation is said to be *folding* along a simple closed curve.

- In the remaining cases, take an orbifold S' with a union l' of one (resp. two) boundary full 1-orbifolds. Take a regular neighborhood N of l' and remove them to obtain V. Identify U with V for $S' - l'$ and $N - l$ to obtain S. Then S is obtained from S' by sewing along l'. Again l corresponds to a full 1-orbifold l'' in S in a one-to-one manner.
- We obtain (5),(6), and (7)-type neighborhoods of l'' in this way, where the operations are said to be *pasting*, *folding*, and *silvering* along full 1-orbifolds respectively.
- In other words, silvering is the operation of removing a regular neighborhood and replacing by a silvered annulus or a half square. Clarifying is an operation of removing the regular neighborhood and replacing an annulus or a silvered strip.

Prop 5.5. The Euler characteristic of an orbifold before and after splitting or sewing remains unchanged.

Proof. Form regular neighborhoods of the involved boundary components of the split orbifold and those of the original orbifold. They have zero Euler characteristic. Since their boundary 1-orbifolds have zero Euler characteristic, the lemma follows by the additivity formula (5.1). □

5.2.7 *Identification interpretations of splitting and sewing*

5.2.7.1 *Identification interpretations of splitting and sewing*

In the following we describe the topological identification process of the underlying space involved in these six types of sewings. The orbifold structure on the sewed orbifold should be clear.

Let an orbifold Σ have a boundary component b. (Σ is not necessarily connected.) b is either a simple closed curve or a full 1-orbifold. We find a 2-orbifold Σ'' constructed from Σ by sewing along b or another component of Σ.

- (A) Suppose that b is diffeomorphic to a circle; that is, b is a closed curve. Let Σ' be a component of the 2-orbifold Σ with boundary component b'. Suppose that there is a diffeomorphism $f : b \to b'$. Then we obtain a bigger orbifold Σ'' glued along b and b' topologically.
  - (I) The construction so that Σ'' does not create any more singular point results in an orbifold Σ'' so that
  $$\Sigma'' - (\Sigma - b \cup b')$$
  is a circle with neighborhood either diffeomorphic to an annulus or a Möbius band.
    - (1) In the first case, $b \neq b'$ (pasting).
    - (2) In the second case, $b = b'$ and $\langle f \rangle$ is of order two without fixed points (cross-capping).

(II) When $b = b'$, the construction so that Σ'' does introduce more singular points to occur in an orbifold Σ'' so that
$$\Sigma'' - (\Sigma - b)$$
is a circle of mirror points or is a full 1-orbifold with endpoints in cone-points of order two depending on whether $f : b \to b$
- (1) is the identity map (silvering), or
- (2) is of order two and has exactly two fixed points (folding).

- (B) Consider when b is a full 1-orbifold with endpoints mirror points.
  - (I) Let Σ' be a component orbifold (possibly the same as one containing b) with boundary full 1-orbifold b' with endpoints mirror points where $b \neq b'$. We obtain a bigger orbifold Σ'' by gluing b and b' by a diffeomorphism $f : b \to b'$. This does not create new singular points (pasting).
  - (II) Suppose that $b = b'$. Let $f : b \to b$ be the attaching map. Then
    - (1) if f is the identity, then b is silvered and the end points are changed into corner-reflectors of order two (silvering).
    - (2) If f is of order two, then Σ'' has a new cone-point of order two and has one-boundary component orbifold removed away. b corresponds to a mixed type 1-orbifold in Σ' (folding).
  - It is obvious how to put the orbifold structure on Σ'' using the previous descriptions using regular neighborhoods above.

5.3 Some helpful references

- S. Choi and W. Goldman, The deformation spaces of convex RP 2 -structures on 2-orbifolds American Journal of Mathematics 127, 5, 1019–1102 (2005)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)

Chapter 6

Geometry of 2-orbifolds

Geometric structures on 2-orbifolds

6.1 Introduction

- Definition of geometric structures on 2-orbifolds
  - Using charts
  - Goodness of geometric 2-orbifolds.
  - Using development pair.
  - Flat X-bundles and transversal sections.
- The deformation spaces of geometric structures on 2-orbifolds
- The local homeomorphism theorem from the deformation space to the representation space.
- The deformation space of (X, G)-structures on an orbifold.
  - Definition
  - The local homeomorphism theorem
    - The isotopy lemma
    - Outline of proof.

%2

6.2 Definition of geometric structures on orbifolds

Let (X, G) be a pair defining a geometry. That is, G is a Lie group acting on a manifold effectively and transitively. Given an orbifold M, there is at least three ways to define (X, G)-geometric structure on M.

- Using atlas of charts.
- A developing map from the universal covering space.
- A cross-section of the flat orbifold X-bundle.

6.2.1 *Atlas of charts approach*

Given an atlas of charts for M, for each chart (U, K, ϕ) we find an X-chart $\rho : U \to X$ and an injective homomorphism $h : K \to G$ so that ρ is an equivariant map. For each imbedding $i : (V, H, \psi) \to (U, K, \phi)$ where V has an X-chart $\rho' : V \to X$ and equivariant with respect to an injective homomorphism $h' : H \to G$, we have

$$\rho \circ i = g \circ \rho', h'(\cdot) = gh(i^*(\cdot))g^{-1}$$

If we simply identify with open subsets of X, the above simplifies greatly and i is a restriction of an element of g and i^* is a conjugation by g also.

This gives us a way to build an orbifold from pieces of X. A maximal such atlas of X-charts is called *an* (X, G)*-structure* on M.

A (X, G)*-map* $M \to N$ is a smooth map f so that for each x and $y = f(x)$, there are charts (U, K, ϕ) and (V, H, ψ) so that f sends $\phi(U)$ into $\psi(V)$ and lifts to $\tilde{f} : U \to V$ so that $\rho' \circ \tilde{f} = g \circ \rho$ and $h'(i^*(\cdot)) = gh(\cdot)g^{-1}$. In otherwards, f is a restriction of an element g of G up to charts with a homomorphism $K \to H$ induced by a conjugation by an element of G.

Theorem 6.2.1. *(X, G)-orbifold is always good.*

Proof. Basically build a germ of local (X, G)-maps from M to X which is a principal bundle and is a manifold: For each (U, K, ϕ), we build $G(U) = G \times U/K$ and a projection $G(U) \to U$. We paste these together to find $G(M)$. Then $G(M)$ is a manifold since K acts on $G \times U$ freely. The foliation given by pasting $g_0 \times U$ is a foliation by open manifolds with the same dimension as M. Each leave of the foliation is covers M. □

If G is a subrgroup of a linear group, then M is very good by Selberg's lemma. Thus M is a quotient $\tilde{M}/\Gamma$ where Γ is finite and contains copies of all of the local group.

6.2.2 *The developing maps and holonomy homomorphisms*

Let $\tilde{M}$ denote the universal cover of M with a deck transformation group π. Then we obtain a *developing map* $D : \tilde{M} \to X$ by first finding an initial chart $\rho : U \to X$ and continuing by extending maps by patches. One uses a nice cover of $\tilde{M}$ and extend. The map is well-defined independently of which path of charts one took to arrive at a given chart. To show this, we need to homotopy and consider three nice charts simultaneously and the fact that M admits a real analytic structure and the charts are real analytic and hence if they agree on an open set, then they extend each other.

Since we can change the initial chart to $k \circ \rho$ for any $k \in G$, we see that $k \circ D$ is another developing map and conversely any developing map is of such form.

Given a deck transformation $\gamma : \tilde{M} \to \tilde{M}$, we see that $D \circ \gamma$ is a developing map also and hence equals $h(\gamma) \circ D$ for some $h(\gamma) \in G$.

The map $h : \pi \to G$ is a homomorphism, so-called the holonomy homomorphism.

The pair (D, h) is said to be the *development pair*. The development pair is determined up to an action of G given by $(D, h(\cdot)) \to (g \circ D, gh(\cdot)g^{-1})$.

Conversely, a developing map (D, h) gives us X-charts: For each open chart (U, K, ψ), we lift to a component of $p^{-1}(U)$ in $\tilde{M}$ and obtain a restriction of D to the component. This gives us X-charts. A different choice of components gives us the compatible charts. Local group actions and imbeddings satisfy the desired properties. Thus, a development pair completely determines the (X, G)-structure on M.

More details here? Is this somewhat repeated?

6.2.3 *Definition as flat bundles with sections*

Given an (X, G)-manifold with X-charts, form a G-bundle $G(M)$ as above. This is a principal G-bundle. We form an associated an X-bundle $X(M)$ using the G-action on X. $X(M) = G(M) \times X/G$ where G acts on the right on $G(M)$ and left on X. and G acts on $G(M) \times X$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in G(M), x \in X.$$

A flat G-bundle is an object obtained by patching open sets $G \times U$ by the left action of G, and so is a flat X-bundle

6.2.3.1 *Flat X-bundles*

A foliation in $G(M)$ induces a foliation in $G(M) \times X$ and hence a foliation in $X(M)$ transversal to fibers. This corresponds to a flat G-connection. A flat G-connection on $X(M)$ is a way to identify each fibers of $X(M)$ with X locally-consistently. A flat G-connection on $X(M)$ gives us a flat G-connection on $X(\tilde{M})$. Since $\tilde{M}$ is a simply-connected manifold, $X(\tilde{M})$

equals $X \times \tilde{M}$ as an X-bundle. $X(\tilde{M})$ covers $X(M)$ and hence $X(M) = X \times \tilde{M}/\pi_1(M)$ where the connection corresponds to foliations with leaves of type $x \times \tilde{M}$, in general.

Hence this gives us a representation $h : \pi_1(M) \to G$ so that for any $\gamma \in \pi_1(M)$, the corresponding action in $X \times \tilde{M}$ is given by $(x, m) \to (h(\gamma)x, \gamma(x))$. Conversely, given a representation h, we can build $X \times \tilde{M}$ and act by $\gamma(x, m) = (h(\gamma), \gamma(m))$ to obtain a flat X-bundle $X(M)$.

6.2.3.2 *Flat X-bundles with sections*

Conversely, a development pair gives us a flat X-bundle $X(M)$ with a section $s; M \to X(M)$. We obtain a section $D' : \tilde{M} \to X \times \tilde{M}$ transversal to the foliation by taking $D'(x) = (D(x), x)$ for $x \in \tilde{M}$. The transversality D' to the constant foliation is actually equivalent to the immersive property of D.

The left-action of $\pi_1(M)$ gives us a section $s : M \to X(M)$ transversal to the foliation.

On the other hand, given a transversal section $s : M \to X(M)$, we obtain a transversal section $s' : \tilde{M} \to X \times \tilde{M}$. By a projection to X, we obtain an immersion $D : \tilde{M} \to X$ so that $D \circ \gamma = h(\gamma) \circ D$ for some $h(\gamma)$ in G. The map $h : \pi_1(M) \to G$ is a homomorphism. Hence we obtain a development pair.

6.2.4 *The equivalences of three notions.*

Given an atlas of X-charts, i.e., a (X, G)-structure, we determine a development pair (D, h). Given a development pair (D, h), we determine an atlas of X-charts, i.e., an (X, G)-structure. Given a development pair (D, h), we determine a flat X-bundle $X(M)$ with a transversal section $M \to X(M)$. Given a section $s : M \to X(M)$ to a flat X-bundle, we determine a development pair (D, h). Thus, these three class of defintions are equivalent.

6.3 Definition of the deformation space of (X, G)-structures on orbifolds

Consider the set $\mathcal{M}(M)$ of all (X, G)-structures on an orbifold M. We introduce an equivalence relation $\sim$: two (X, G)-structures μ_1 and μ_2 are equivalent if there is an isotopy $\phi : M \to M$ so that $\phi^*(\mu_1) = \mu_2$. The deformation space of (X, G)-structures on M is $\mathcal{M}/\sim$. We reinterpret the space as

- The set of diffeomorphisms $f : M \to M'$ for M an orbifold and M' an (X, G)-orbifold.
- The equivalence relation $f : M \to M'$ and $g : M \to M"$ if exists an (X, G)-diffeomorphism $h : M' \to M"$ so that $h \circ f$ is isotopic to g.
- The quotient space is same as above.

6.3.1 *Another interpretations*

First, we identify $\pi_1(M)$ with $\pi_1(M \times I)$. Consider the set of diffeomorphisms $f : \tilde{M} \to \tilde{M}'$ equivariant with respect to isomorphism $f_* : \pi_(M) \to \pi_1(M')$ for an (X, G)-orbifold M'. We introduce an equivalence relation: Given $f : \tilde{M} \to \tilde{M}'$ and $g : \tilde{M} \to \tilde{M}"$, we say that they are equivalent if there exists an (X, G)-map $\phi : \tilde{M}' \to \tilde{M}"$ so that $\phi \circ f$ is isotopic to g by an isotopy $\tilde{M} \times I \to \tilde{M}''$ equivariant with respect to both $\phi_* \circ f_*$ and g_* which are equal. Denote this set by $\mathcal{D}_I(M)$. This set is again one-to-one relation with the above space since we can always lift diffeomorphisms and isotopies.

6.3.1.1 *Isotopy-equivalence space.*

The space $\mathcal{S}(M)$ is defined as follows: Consider the set of $(D, \tilde{f} : \tilde{M} \to \tilde{M}')$ where $f : M \to M'$ is a diffeomorphism for orbifolds M and M' and $D : \tilde{M}' \to X$ is a diffeomorphism equivariant with respect to a homomorphism $h : \pi_1(M') \to G$. Two $(D, \tilde{f})$ and $(D', \tilde{f}' : \tilde{M} \to \tilde{M}")$ are equivalent if there is a diffeomorphism $\phi : M' \to M''$ so that $D' \circ \tilde{\phi} = D$ and an isotopy $H : M \times I \to M''$ equivariant with respect to $\tilde{f}'_* : \pi_1(M) \to \pi_1(M'')$ so that $\phi \circ f = H_0$ and $f' = H_1$. We can finally give topology on this space by C_1 topology using $D \circ \tilde{f}$.

6.3.1.2 *The topology of the deformation space*

Theorem 6.3.1. *There is a natural action of G on $\mathcal{S}(M)$ given by $g(D, \tilde{f}) = (g \circ D, \tilde{f}, g \in G$. The quotient space $\mathcal{D}(M)$ is the deformation space.*

Proof. We show $\mathcal{D}_I(M)$ is one-to-one equivalent to $\mathcal{S}(M)/G$: Given an element $\tilde{f} : \tilde{M} \to \tilde{M}'$, there is a developing map $D : \tilde{M}' \to X$ equivariant with respect to $h : \pi_1(M') \to G$. If $\tilde{f} : \tilde{M} \to \tilde{M}'$ and $\tilde{f}' : \tilde{M} \to \tilde{M}"$ are equivalent, then there is an (X, G)-diffeomorphism $M' \to M$" and hence two global charts D' and D" differ only by an element of G.

Conversely, given $(D, \tilde{f})$, we obviously obtain an (X, G)-structure on M'. If $(D, \tilde{f})$ and $(D', \tilde{f}')$ are equivalent, then there is a diffeomorphism $\phi : M' \to M''$ so that $D' \circ \tilde{\phi} = g \circ D$. This means $\phi' : M' \to M''$ is an (X, G)-diffeomorphism. $\square$

6.3.2 *The local homeomorphism theorem*

6.3.2.1 *The representation space*

Suppose that π is finitely-presented. In particular if M is a compact n-orbifold, this is true. Denote by $g_1, ..., g_n$ the set of generators and $R_1, ..., R_m$ be the set of relations.

The set of homomorphisms $\pi_1(M) \to G$ can be identified with a subset of G^n by sending a homomorphism h to $(h(g_1), ..., h(g_n))$. This clearly injective map. This image can be described as an algebraic subset defined by relations $R_1, ..., R_m$. This follows since if the relation is satisfied, then we can obtain the representation conversely. Denote the space by $Hom(\pi, G)$.

There is an action of G on $Hom(\pi, G)$ given by the action $(g \star h)(\cdot) = gh(\cdot)g^{-1}$ We denote by $Rep(\pi, G)$ the quotient space $Hom(\pi, G)/G$.

6.3.2.2 *The map hol*

We can define $hol' : \mathcal{S}(M) \to Hom(\pi, G)$. This induces $hol : \mathcal{M}_{X,G}(M) \to Rep(\pi, G)$. The main purpose of this section is to state:

Theorem 6.3.2. *hol is a local homeomorphism.*

Proof. We send $(D, \tilde{f})$ to the associated homomorphism $h : \pi \to G$. First, it is easy to show that *hol* is continuous: If $D' \circ \tilde{f}'$ is sufficiently close to $D \circ f$ in a sufficiently large compact subset of $\tilde{M}$, then the holonomy $h'(g_i)$ of generators g_i is as close to the original $h(g_i)$ as possible.

The converse is given in the subsubsections below. The idea is to find a geometric structure corresponding to h and if one deforms h by a small amount, then we can change the geometric structure correspondingly by considering local models and changing them each by using Lemma **??** and patching up the differences in a consistent way. Finally, we have to show that such change of geometric structures is unique up to isotopies. $\square$

The proof is shortened considerably. See [**(author?)** (Choi)] for details.

The local homeomorphism result was very important for the study of deformations of (X, G)-structures on manifolds, introduced by Weil [**?**)]. The same can be said for orbifolds. For manifolds, Thurston gave a proof (see [**?**)]). Later J. Morgan gave a lecture of it, which is written up by Lok [**?**)] in his Ph.D. thesis. Also, Canary and Epsten gave a proof of it also. [**?**)].

6.3.2.3 *The stable representations*

There is a dense open subset, called the *stable subset*, of $Hom(\pi, G)$ where G acts properly. Denote this space by $Hom^s(\pi, G)$ and its quotient by $Rep^s(\pi, G)$.

If we denote by $\mathcal{D}^s(M)$ the subset of $\mathcal{D}$ whose holonomies are in the stable region. Then there is a local homeomorphism $\mathcal{D}^2(M) \to Rep^s(\pi, G)$ since the right action on developing map gives a conjugation action on holonomy homomorphisms.

6.4 Notes

The main part of this chapter is from [**(author?)** (Choi)] and [**(author?)** (Choi and Goldman)]. [**(author?)** (Kapovich)] also devotes some pages to geometric orbifolds. (See also [**?**)]).

Chapter 7

Deformation spaces of hyperbolic structures on 2-orbifolds

Teichmüller spaces of 2-orbifolds

7.1 Introduction

- The definition of the Teichmüller space of 2-orbifolds
- The geometric cutting and pasting and the deformation spaces
- The decomposition of 2-orbifolds into elementary orbifolds.
- The Teichmüller spaces of 2-orbifolds

7.2 The definition of the Teichmüller space of 2-orbifolds

A *hyperbolic* structure on a 2-orbifold is a geometric structure modeled on H^2 with the isometry group $\mathrm{PSL}(2, \mathbb{R})$. The *Teichmüller* space $\mathcal{T}(M)$ of a 2-orbifold M is the deformation space of hyperbolic structures on the 2-orbifold. As before, we reinterpret the space as

- The set of diffeomorphisms $f : M \to M'$ for M an orbifold and M' a hyperbolic 2-orbifold.
- The equivalence relation $f : M \to M'$ and $g : M \to M$" if exists a hyperbolic isometry $h : M' \to M$" so that $h \circ f$ is isotopic to g.
- The quotient space is same as above.

A necessary condition for an orbifold to have a hyperbolic structure is that the orbifold euler characteristic be negative: This follows from the Gauss-Bonnet theorem. Here the negative of the hyperbolic area is the Euler characteristic times 2π.

A closed 2-orbifold with a complex structure has a unique hyperbolic structure provided it is compact and has negative Euler characteristic. The deformation space of complex structures on a closed 2-orbifold is identical with the Teichmuller space as defined here by the uniformization theorem.

7.3 The geometric cutting and pasting and the deformation spaces

A compact geodesic 1-orbifold without boundary points in the interior of a 2-orbifold Σ are either

- a closed geodesic in the interior or entirely in the boundary of $|\Sigma|$ or
- a segment with two silvered points which are either at silvered edges or cone-points of order two. The topological interior is either in the interior or entirely in the boundary of $|\Sigma|$.

The geometric type is classified by length and the topological type. Such a geodesic 1-orbifold is covered by a closed geodesic in some cover of the 2-orbifold, which is a surface.

The *Teichmüller space* $\mathcal{T}(I)$ for a 1-orbifold I is the product of the space of lengths $\mathbb{R}^+$s for each component of I.

7.3.1 *Geometric constructions.*

Recall the type of topological constructions with 1-orbifolds. Suppose they are boundary components of 2-orbifolds whose components have negative Euler characteristics.

- (A)(I) Pasting or crosscapping along simple closed curves.
- (A)(II) Silvering or folding along a simple closed curve.
- (B)(I) Pasting along two full 1-orbifolds.
- (B)(II) Silvering or folding along a full 1-orbifold.

Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric version of the above.

Suppose that the involved 1-orbifolds are geodesic boundary components of a hyperbolic 2-orbifolds.

- (A)(I). For pasting two closed geodesics, we have a $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. (Here the length of two closed geodsics have to be the same.)

- (A)(I) For cross-capping, we have a unique isometry. The isometry has to be a slide reflection of distance half the length of the closed geodesic. (There is no conditions.)

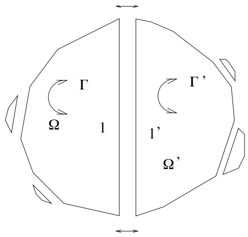


Fig. 7.1 The actions here are isometries on $\mathbb{R}^2$.

- (A)(II). For folding a closed geodesics, we have a $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The choice depends on the choice of two fixed points of the pasting map. The distance is the half of length of the closed geodesic. (no condition)
- (A)(II) For silvering, we have unique isometry to do this. (no condition)

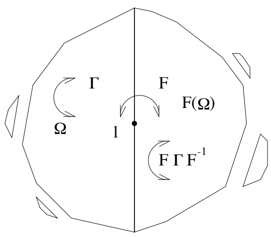


Fig. 7.2 The actions here are isometries on $\mathbb{R}^2$.

- (B)(I). For pasting along two geodesic full 1-orbifolds, We have a unique way to do this. The lengths of the orbifolds have to be the same.

- (B)(II) For silvering and folding, we have unique isometry to do this. (no condition)

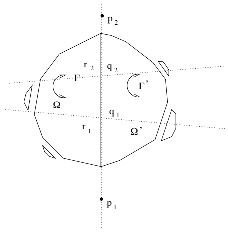


Fig. 7.3 The actions here are isometries on $\mathbb{R}^2$.

7.3.2 *Teichmuller spaces under the geometric operations*

(A)(I)(1) Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration, where Δ is the subset of $\mathcal{T}(\Sigma')$ where b and b' have equal legnths.

(A)(I)(2) Let Σ'' be obtained from Σ' by cross-capping. The resulting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

(A)(II)(1) Let Σ'' be obtained from Σ' by silvering. The clarifying map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

(A)(II)(2) Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration.

(B)(I) Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a diffeomorphism where Δ is a subset of $\mathcal{T}(\Sigma')$ where the lengths of b and b' are equal.

(B)(II) Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

7.4 The decomposition of 2-orbifolds into elementary orbifolds.

7.4.1 *Topological decomposition of hyperbolic 2-orbifolds into elementary orbifolds along geodesic 1-orbifolds.*

Suppose that Σ is a compact hyperbolic orbifold with $\chi(\Sigma) < 0$ and geodesic boundary. Let $c_1, \dots, c_n$ be a mutually disjoint collection of simple closed curves or 1-orbifolds so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \cdots - c_n$ is negative. Then $c_1, \dots, c_n$ are isotopic to simple closed geodesics or geodesic full 1-orbifolds $d_1, \dots, d_n$ respectively where $d_1, \dots, d_n$ are mutually disjoint.

7.4.1.1 *Elementary 2-orbifolds.*

We require the boundary components be geodesics.

(P1) A pair-of-pants.
(P2) An annulus with one cone-point of order n. ($A(; n)$)
(P3) A disk with two cone-points of order p, q, one of which is greater than 2. ($D(; p, q)$)
(P4) A sphere with three cone-points of order p, q, r where $1/p+1/q+1/r < 1$. ($\mathbf{S}^2(; p, q, r)$)

(A1) An annulus with one boundary component a union of a singular segment and one boundary-orbifold. (*2-pronged crown* and $A(2, 2;)$.) It has two corner-reflectors of order 2 if the boundary components are silvered.
(A2) An annulus with one boundary component of the underlying space in a singular locus with one corner-reflector of order n, $n \geq 2$. (The other boundary component is a closed geodesic which is the boundary of the orbifold.) (We call it a *one-pronged crown* and denote it $A(n;)$.)

(A3) A disk with one singular segment and one boundary 1-orbifold and a cone-point of order greater than or equal to three ($D^2(2, 2; n)$).

(A4) A disk with one corner-reflector of order m and one cone-point of order n so that $1/2m + 1/n < 1/2$ (with no boundary orbifold). ($n \geq 3$ necessarily.) ($D^2(m; n)$.)

(D1) A disk with three edges and three boundary 1-orbifolds. No two boundary 1-orbifolds are adjacent. (We call it a *hexagon* or $D^2(2, 2, 2, 2, 2, 2;)$.)

(D2) A disk with three edges and two boundary 1-orbifolds on the boundary of the underlying space. Two boundary 1-orbifolds are not adjacent, and two edges meet in a corner-reflector of order n, and the remaining one a segment. (We called it a *pentagon* and denote it by $D^2(2, 2, 2, 2, n;)$.)

(D3) A disk with two corner-reflectors of order p, q, one of which is greater than or equal to 3, and one boundary 1-orbifold. The singular locus of the disk is a union of three edges and two corner-reflectors. (We call it a *quadrilateral* or $D^2(2, 2, p, q;)$.)

(D4) A disk with three corner-reflectors of order p, q, r where $1/p + 1/q + 1/r < 1$ and three edges (with no boundary orbifold). (We call it a *triangle* or $D^2(p, q, r;)$.)

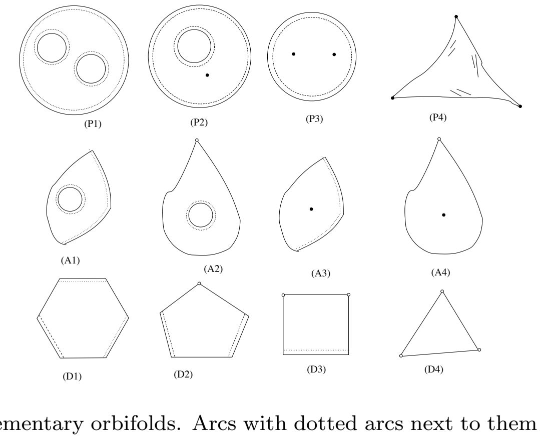


Fig. 7.4 The elementary orbifolds. Arcs with dotted arcs next to them indicate boundary components. Black points indicate cone-points and white points the corner-reflectors.
.

7.4.1.2 *The geometric decomposition into elementary orbifolds*

Let Σ be a compact hyperbolic orbifold with $\chi(\Sigma) < 0$ and geodesic boundary. Then there exists a mutually disjoint collection of simple closed geodesics and mirror- or cone- or mixed-type geodesic 1-orbifolds so that Σ decomposes along their union to a union of elementary 2-orbifolds or such elementary 2-orbifolds with some boundary 1-orbifolds silvered additionally.

7.4.2 *The Teichmüller spaces for 2-orbifolds*

Theorem 7.4.1. *Thurston's theorem*

Let Σ be a compact 2-orbifold with empty boundary and negative Euler characteristic diffeomorphic to an elementary 2-orbifold. Then the deformation space $\mathcal{T}(\Sigma)$ of hyperbolic $\mathbb{RP}^2$-structures on Σ is homeomorphic to a cell of dimension $-3\chi(X_\Sigma) + 2k + l + 2n$ where X_Σ is the underlying space and k is the number of cone-points, l is the number of corner-reflectors, and n is the number of boundary full 1-orbifolds.

7.4.2.1 *Strategy of proof*

Prop 7.1. Proposition A: for each elementary 2-orbifold S, $\mathcal{T}(S)$ is homeomorphic to $\mathcal{T}(\partial S)$, where $\mathcal{T}(\partial S)$ is the product of $\mathbb{R}^+$ for each component of ∂S corresponding to the hyperbolic-metric lengths of components of ∂S. Then for hyperbolic structures, to obtain a bigger orbifold, we need to use the above result about the Teichmüller spaces under geometric decompositions.

7.4.2.2 *The generalized hyperbolic triangle theorem*

A generalized triangle in the hyperbolic plane is one of following:

(a) A hexagon: a disk bounded by six geodesic sides meeting in right angles labeled $A, \beta, C, \alpha, B, \gamma$.
(b) A pentagon: a disk bounded by five geodesic sides labeled A, B, α, C, β where A and B meet in an angle γ, and the rest of the angles are right angles.
(c) A quadrilateral: a disk bounded by four geodesic sides labeled A, C, B, γ where A and C meet in an angle β, C and B meet in an angle α and the two remaining angles are right angles.

(d) A triangle: a disk bounded by three geodesic sides labeled A, B, C where A and B meet in an angle γ and B and C meet in an angle α and C and A meet in angle β.

7.4.2.3 *The generalized hyperbolic triangles*

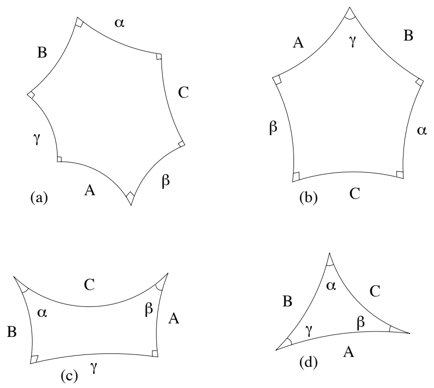


Fig. 7.5 .

7.4.2.4 *The trigonometry*

For generalized triangles in the hyperbolic plane,

$$\begin{aligned}
&\text{(a) } \cosh C = \frac{\cosh\alpha\cosh\beta + \cosh\gamma}{\sinh\alpha\sinh\beta}\\
&\text{(b) } \cosh C = \frac{\cosh\alpha\cosh\beta + \cos\gamma}{\sinh\alpha\sinh\beta}\\
&\text{(c) } \sinh A = \frac{\cosh\gamma\cos\beta + \cos\alpha}{\sinh\beta\sin\gamma}\\
&\text{(d) } \cosh C = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}
\end{aligned} \tag{7.1}$$

In (a), (α, β, γ) can be any positive numbers. In (b), (α, β) can be any positive numbers and γ in $(0, \pi)$ In (c), (α, β) can be any positive real numbers in $(0, \pi)$ satisfying $\alpha + \beta < \pi$, and γ any real number. In (d), (α, β, γ) can be any real numbers in $(0, \pi)$ satisfying $\alpha + \beta + \gamma < \pi$.

7.4.2.5 *The proof of Proposition A.*

The following lemmas imply Proposition A

Lemma 7.4.2. *For elementary* 2*-orbifolds of type (D1), (D2), (D3), and (D4). Silvered edges are labeled by the capital letters* A, B, C*. Assign to each vertex an angle of the form* π/n *(where* ($n > 1$ *is an integer*)*, for which it is a corner-reflector of that angle. Each edge labeled by Greek letters* α, β, γ *is a boundary full* 1*-orbifold. Then in cases* (a), (b), (c), (d) $\mathcal{F} : \mathcal{T}(P) \to \mathcal{T}(\partial P)$ *for each of the above orbifolds* P *is a homeomorphism*; *that is,* $\mathcal{T}(P)$ *is homeomorphic to a cell of dimension* 3*,* 2*,* 1*, or* 0 *respectively.*

Lemma 7.4.3. *Let* S *be an elementary* 2*-orbifold of type* (A1), (A2), (A3)*, or* (A4)*. Then* $\mathcal{F} : \mathcal{T}(S) \to \mathcal{T}(\partial S)$ *is a homeomorphism. Thus,* $\mathcal{T}(S)$ *is a cell of dimension* $2, 1, 1,$ *or* 0 *when* S *is of type* (A1), (A2), (A3) *or* (A4) *respectively. In case* (A4), $\mathcal{T}(S)$ *is a single point.*

For elementary orbifolds of type (P1),(P2),(P3), or (P4), we simply notices that they double covers orbifolds of type (D1),(D2),(D3), or (D4) which is realized as isometries where each of the boundary components do the same. In fact, the isometry can be explictly constructed by taking shortest geodesics between boundary components.

7.4.2.6 *The steps to prove Theorem A.*

Let a 2-orbifold Σ, each component of which has negative Euler characteristic, be in a *class* $\mathcal{P}$ if the following hold:

(i) The deformation space of hyperbolic $\mathbb{RP}^2$-structures $\mathcal{T}(\Sigma)$ is diffeomorphic to a cell of dimension

$$-3\chi(X_\Sigma) + 2k + l + 2n$$

where k is the number of cone-points, l the number of corner-reflectors, n is the number of boundary full 1-orbifolds.

(ii) There exists a principal fibration

$$\mathcal{F} : \mathcal{T}(\Sigma) \to \mathcal{T}(\partial\Sigma)$$

with the action by a cell of dimension $\dim \mathcal{T}(\Sigma) - \dim \mathcal{T}(\partial\Sigma)$.

Let Σ be a 2-orbifold whose components are orbifolds of negative Euler characteristic, and it splits into an orbifold Σ' in $\mathcal{P}$. We suppose that (i) and (ii) hold for Σ', and show that (i) and (ii) hold for Σ. Since Σ eventually decomposes into a union of elementary 2-orbifolds where (i) and (ii) hold, we would have completed the proof.

The proof follows by going through each of the constructions....

(A)(I)(1) Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \Delta \subset \mathcal{T}(\Sigma')$$
is a principal $\mathbb{R}$-fibration, where Δ is the subset of $\mathcal{C}(\Sigma')$ where b and b' have equal invariants.

(A)(I)(2) Let Σ'' be obtained from Σ' by cross-capping. The resulting map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$
is a diffeomorphism.

(A)(II)(1) Let Σ'' be obtained from Σ' by silvering. The clarifying map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$
is a diffeomorphism.

(A)(II)(2) Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$
is a principal $\mathbb{R}$-fibration.

(B)(I) Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \Delta \subset \mathcal{T}(\Sigma')$$
is a diffeomorphism where Δ is a subset of $\mathcal{T}(\Sigma')$ where the invariants of b and b' are equal.

(B)(II) Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map
$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$
is a diffeomorphism.

7.5 Some helpful references

These theory were created in [Thurston (10)] and were written in [**(author?)** (Matsumoto and Montesinos-Amilibia)] and [**(author?)** (Ohshika)]. (See also [**(author?)** (Kapovich)].) The materials here are from [**(author?)** (Choi)] and [**(author?)** (Choi and Goldman)].

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