UNAVOIDABLE INDUCED SUBGRAPHS IN LARGE GRAPHS WITH NO HOMOGENEOUS SETS

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Abstract. An n-vertex graph is *prime* if it has no non-trivial *homogeneous* set, that is a set X of vertices ($2 \leq |X| \leq n-1$) such that every vertex not in X is either complete or anticomplete to X. A *chain* of length t is a sequence of $t+1$ vertices such that for every vertex in the sequence except the first one, its immediate predecessor is its unique neighbor or its unique non-neighbor among all of its predecessors. We prove that for all n, there exists N such that every prime graph with at least N vertices contains one of the following graphs or their complements as an induced subgraph: (1) the graph obtained from $K_{1,n}$ by subdividing every edge once, (2) the line graph of $K_{2,n}$, (3) the line graph of the graph in (1), (4) the half-graph of height n, (5) a prime graph induced by a chain of length n, (6) two particular graphs obtained from the half-graph of height n by making one side a clique and adding one vertex.

1. Introduction

All graphs in this paper are simple and undirected. We wish to prove a theorem analogous to the following theorems. (For missing definitions in the following, please refer to the referenced papers.)

Ramsey's theorem: *Every sufficiently large graph contains K_n or $\overline{K_n}$ as an induced subgraph.*

folklore; see Diestel [1, Proposition 9.4.1]**:** *Every sufficiently large* ***connected*** *graph contains K_n, $K_{1,n}$, or a path of length n as an induced subgraph.*

folklore; see Diestel [1, Proposition 9.4.2]**:** *Every sufficiently large* ***2-connected*** *graph contains C_n or $K_{2,n}$ as a topological minor.*

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Oporowski, Oxley, and Thomas [5]**:** *Every sufficiently large **3-connected** graph contains the n-spoke wheel or $K_{3,n}$ as a minor.*

Oporowski, Oxley, and Thomas [5]**:** *Every sufficiently large **internally 4-connected** graph contains the $2n$-spoke double wheel, the n-rung circular ladder, the n-rung Möbius ladder, or $K_{4,n}$ as a minor.*

Ding, Chen [2]**:** *Every sufficiently large **connected and anticonnected** graph contains one of the following graphs or their complements as an induced subgraph: a path of length n, the graph obtained from $K_{1,n}$ by subdividing an edge once, $K_{2,n}$ minus one edge, or the graph obtained from $K_{2,n}$ by adding an edge between two degree-n vertices x_1 and x_2 and adding a pendant edge at each x_i.*

Kwon, Oum [3]**:** *Every sufficiently large graph with **no non-trivial split** contains, as a vertex-minor, a cycle of length n or the line graph of $K_{2,n}$.*

These results state that every sufficiently large graph satisfying certain connectivity requirements contains at least one of the given graphs. Furthermore, in all these theorems, the list is best possible in the sense that each given graph satisfies the required connectivity, can grow its size arbitrary, and does not contain other given graphs in the list.

In this paper, we focus on graphs with no non-trivial homogeneous sets. A set X of vertices of a graph is *homogenenous* if every vertex outside of X is either adjacent to all vertices in X or adjacent to no vertex in X. In the literature, homogeneous sets are also called *modules*, *partitive sets*, *autonomous sets*, and various other terms [4]. We say that a homogeneous set of a graph G is *trivial* if $|X| \leq 1$ or $X = V(G)$ and *non-trivial* otherwise. A graph is called *prime* if every homogeneous set in it is trivial.

We are interested in unavoidable induced subgraphs in prime graphs. Sumner [6, 7] showed the following easy theorem.

Theorem 1.1 (Sumner [6, 7])**.** *Every prime graph with at least 3 vertices has an induced path of length*[1] *3.*

What can we say for large prime graphs? Before stating our main theorem, we list some basic graph classes. For a set X of vertices, we say that X is a *clique* if every pair of distinct vertices in X is adjacent

[1]A *length* of a path is its number of edges.

and we say X is *independent* if every pair of vertices in X is non-adjacent. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set.

- The *half-graph* H_n of height n is a bipartite graph on $2n$ vertices a_1, a_2, …, a_n, b_1, b_2, …, b_n such that a_i is adjacent to b_j if and only if $i \geq j$.
- The *half split graph* H'_n of height n is the graph obtained from H_n by making $\{b_1, b_2, \ldots, b_n\}$ a clique. Let $H'_{n,I}$ be the graph obtained from H'_n by adding a new vertex adjacent to $a_1, a_2, \ldots, a_n$. Let H^*_n be the graph obtained from H'_n by adding a new vertex adjacent to a_1.
- The *thin spider* with n legs is a split graph on $2n$ vertices consisting of an independent set $\{a_1, a_2, \ldots, a_n\}$ and a clique $\{b_1, b_2, \ldots, b_n\}$ such that a_i is adjacent to b_j if and only if $i = j$. Thus, the thin spider with n legs is isomorphic to the line graph of $K^{(1)}_{1,n}$. The *thick spider* with n legs is a split graph on $2n$ vertices consisting of an independent set $\{a_1, a_2, \ldots, a_n\}$ and a clique $\{b_1, b_2, \ldots, b_n\}$ such that a_i is adjacent to b_j if and only if $i \neq j$. So, a thick spider is the complement of a thin spider. A *spider* is a thin spider or a thick spider.

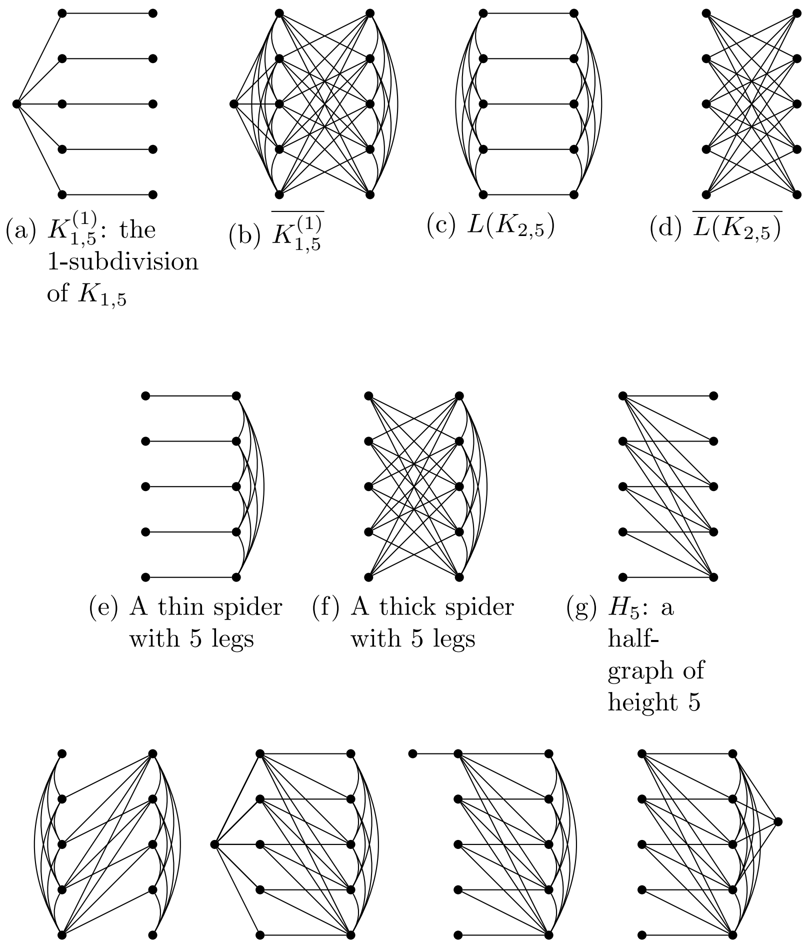
A sequence $v_0, v_1, \ldots, v_n$ of distinct vertices of G is called a *chain* from a set $I \subseteq V(G)$ to v_n if $n \geq 2$, $v_0, v_1 \in I$, $v_2, v_3, \ldots, v_n \notin I$, and for $i > 0$, v_{i-1} is either the unique neighbor or the unique non-neighbor of v_i in $\{v_0, v_1, \ldots, v_{i-1}\}$. The *length* of a chain $v_0, v_1, \ldots, v_n$ is n. For example, a sequence of vertices inducing a path of length t is a chain of length t.

Here is our main theorem.

Theorem 1.2. *For every integer $n \geq 3$, there exists N such that every prime graph with at least N vertices contains one of the following graphs or their complements as an induced subgraph.*

(1) The 1-subdivision of $K_{1,n}$ (denoted by $K^{(1)}_{1,n}$).
(2) The line graph of $K_{2,n}$.
(3) The thin spider with n legs.
(4) The half-graph of height n.
(5) The graph $H'_{n,I}$.
(6) The graph H^*_n.
(7) A prime graph induced by a chain of length n.

Note that all graphs in (1)–(7) are prime. It is straightforward to prove that none of the graphs in the list can be omitted, by showing



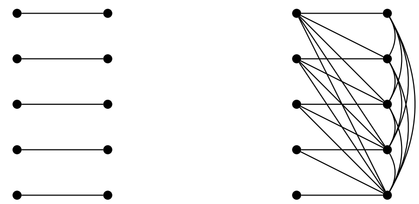
(a) $K_{1,5}^{(1)}$: the 1-subdivision of $K_{1,5}$
(b) $\overline{K_{1,5}^{(1)}}$
(c) $L(K_{2,5})$
(d) $\overline{L(K_{2,5})}$
(e) A thin spider with 5 legs
(f) A thick spider with 5 legs
(g) H_5: a half-graph of height 5
(h) $\overline{H_5}$
(i) $H'_{5,I}$
(j) H_5^*
(k) $\overline{H_5^*}$

FIGURE 1. Typical prime graphs

that none of the graphs in the list contains a large graph of other types or the complement of large graphs in the list.

2. CHAIN

A path is an object to certify connectedness and a chain can be used analogously to certify primeness. In the next proposition, we will prove that in a graph G, there is a chain from every set of two vertices to every other vertex if and only if G is prime. We say that for a set X of vertices, a vertex $v \notin X$ is *mixed* on X if v has both a neighbor and a non-neighbor in X.



(a) $5K_2$ (b) H_5': the half split graph of height 5

FIGURE 2. Some non-prime graphs

Proposition 2.1. *Let $I \subseteq V(G)$ be a set with at least two vertices. and let v be a vertex in $V(G) \setminus I$. Then G has a chain from I to v if and only if all homogeneous sets containing I as a subset contain v.*

Proof. For the forward direction, let X be a set such that $I \subseteq X$ and $v \notin X$. Let $v_0, v_1, \ldots, v_n = v$ be a chain. Let i be the minimum positive integer such that $v_i \notin X$. Then $i \geq 2$ because $|I| \geq 2$. Since v_i is mixed on $\{v_0, v_1, \ldots, v_{i-1}\} \subseteq X$, X cannot be homogeneous.

For the backward direction, let us construct an auxiliary digraph H on $\{\eta\} \cup (V(G) \setminus I)$. For a vertex $w \in V(G) \setminus I$, (η, w) is an arc of H if and only if w is mixed on I. For two distinct vertices $x, y \in V(G) \setminus I$, (x, y) is an arc of H if and only if y is not mixed on I and y is mixed on $I \cup \{x\}$.

Suppose that all homogeneous sets containing I as a subset contain v and v cannot be reached from η by a directed path in H. Let Z be the set of vertices that can be reached from η by a directed path in H. If $y \in V(G) \setminus Z$, then y is not mixed on I, as (η, y) is not an arc of H. Furthermore, for each $x \in Z \setminus \{\eta\}$, as (x, y) is not an arc of H, y is not mixed on $I \cup \{x\}$. Therefore, $Z' = (Z \setminus \{\eta\}) \cup I$ is homogenenous in G, contradicting our assumption that there is no homogeneous set containing I but not containing v.

Thus, H has a shortest directed path P from η to v with vertices η, w_1, w_2, $\ldots$, $w_n = v$ in order. Let v_0, v_1 be a neighbor and a non-neighbor of w_1 in I, respectively. Such a choice exists because w_1 is mixed on I. It follows easily that the sequence

$$v_0, v_1, w_1, w_2, \ldots, w_n$$

is a chain from I to v. □

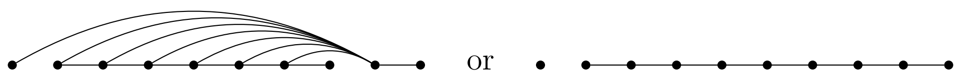


FIGURE 3. Two non-prime graphs induced by chains

Not every chain induces a prime graph. For instance, the graphs in Figure 3 are induced by chains but are not prime. In the following, we characterize which chains induce prime subgraphs.

Proposition 2.2. *Let $t \geq 3$ and $C = v_0, v_1, \ldots, v_t$ be a chain of length t in G. Then C induces a prime subgraph of G if and only if each of v_0 and v_1 have a neighbor other than v_{t-1} and a non-neighbor other than v_{t-1} in the chain.*

This proposition implies that if a graph G is induced by a chain and is not prime, then G is isomorphic to one of the graphs in Figure 3 and their complements.

Proof. We may assume that $V(G) = \{v_0, v_1, \ldots, v_t\}$. Note that we may swap v_0 and v_1 and still obtain a chain.

Let us first prove the forward direction. Suppose that G is prime. By taking the complement and swapping v_0 and v_1 if necessary, we may assume that v_0 has no neighbor other than v_{t-1}. If v_0 is non-adjacent to v_{t-1}, then $\{v_1, v_2, \ldots, v_t\}$ is homogeneous. Therefore v_0 is adjacent to v_{t-1}. It follows that v_{t-1} is the unique neighbor of v_t and so $\{v_0, v_t\}$ is homogeneous. This contradicts our assumption that G is prime.

For the backward direction, suppose that each of v_0 and v_1 have a neighbor other than v_{t-1} and a non-neighbor other than v_{t-1} and furthermore G has a non-trivial homogeneous set X. By Proposition 2.1, either $v_0 \notin X$ or $v_1 \notin X$, because otherwise $V(G) \subseteq X$. We may assume that $v_0 \notin X$ by swapping v_0 and v_1 if necessary.

We may assume that v_t is non-adjacent to v_0 by taking the complement graph if needed. Thus v_{t-1} is the unique neighbor of v_t.

For $0 \leq i < j < t$, the sequence $v_i, v_j, v_{j+1}, v_{j+2}, \ldots, v_t$ is a chain and by Proposition 2.1, if $v_i, v_j \in X$, then $v_{j+1}, v_{j+2}, \ldots, v_t \in X$. Since $|X| \geq 2$, $v_t \in X$.

We claim that for each $a \in \{1, 2, 3, \ldots, t-2\}$, if $v_a \notin X$, then $v_{a+1} \notin X$. Suppose not. Then v_0, v_a are non-adjacent to $v_t \in X$ and so they are non-adjacent to $v_{a+1} \in X$, contradicting the definition of a chain. This implies that $v_1 \in X$ because $|X| \geq 2$.

If $v_2 \in X$, then $v_3, v_4, \dots, v_t \in X$ because $v_1 \in X$. Since v_0 is non-adjacent to v_t, no neighbor of v_0 is in X, contradicting our assumption that v_0 has a neighbor other than v_{t-1}. Thus, $v_2 \notin X$ and so $v_2, v_3, \dots, v_{t-1} \notin X$.

So $X = \{v_1, v_t\}$. However, v_{t-1} is the unique neighbor of v_t and v_1 has a neighbor other than v_{t-1} and therefore X is not homogeneous. □

Corollary 2.3. *Let $t > 3$. Every chain of length t contains a chain of length $t-1$ inducing a prime subgraph.*

Proof. Let $v_0, v_1, \dots, v_t$ be a chain and let G be the graph induced on $\{v_0, v_1, \dots, v_t\}$. By taking the complement and swapping v_0 and v_1, we may assume that v_t is complete to $\{v_0, v_1, v_2\}$ and v_0 is adjacent to v_2. Now consider the chain $v_1, v_2, \dots, v_t$. We know that v_1 and v_2 are non-adjacent and v_t is complete to $\{v_1, v_2\}$ which implies that $v_1, v_2, \dots, v_t$ is a chain inducing a prime subgraph by Proposition 2.2. □

3. Prime graphs containing a large independent set

If a prime graph G is very large, then it will have a large independent set or a large clique by Ramsey's Theorem. If it has a large clique, then we may take the complement graph and assume that G has a large independent set, because the complement of G is also prime. In this section, we apply Ramsey's Theorem to extract some induced subgraphs that are not necessarily prime. Later sections will be devoted to growing these subgraphs into prime subgraphs.

Proposition 3.1. *For all integers $n, n_1, n_2 > 0$, there exists $N = f(n, n_1, n_2)$ such that every prime graph with an N-vertex independent set contains an induced subgraph isomorphic to*

1. *a spider with n legs,*
2. *$\overline{L(K_{2,n})}$,*
3. *the half-graph of height n,*
4. *the disjoint union of n_1 copies of K_2, denoted by n_1K_2, or*
5. *the half split graph of height n_2.*

Proof. We say that (A, X, Y) is a *regular* triple if A, X, Y are disjoint subsets of vertices such that $A \cup X$ is independent, $|X| = |Y|$, and X and Y can be ordered $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_n\}$ such that for all i, the vertex y_i is either

1. adjacent to x_i and anti-complete to $\{x_{i+1}, x_{i+2}, \dots, x_n\} \cup A$, or
2. non-adjacent to x_i and complete to $\{x_{i+1}, x_{i+2}, \dots, x_n\} \cup A$.

(We allow y_i to be adjacent to x_j if $j < i$.)

We claim that if (A, X, Y) is a regular triple of a prime graph G with $1 < |A| < |V(G)|$, then there is a regular triple (A', X', Y') such that $A' \subseteq A$, $|A'| \geq |A|/2$ and $|X'| = |Y'| = |X|+1 = |Y|+1$. Since A is not homogeneous, there is a vertex y such that y has both a neighbor and a non-neighbor in A. Since $A \cup X$ is independent, y does not belong to X. Since each y_i is complete or anti-complete to A, y is not in Y. Let A_y be the set of all neighbors of y in A. If $|A_y| \geq |A|/2$, then $(A_y, X \cup \{x\}, Y \cup \{y\})$ is a regular triple for a non-neighbor x of y in A. If $|A_y| < |A|/2$, then $(A \setminus A_y, X \cup \{x\}, Y \cup \{y\})$ is a regular triple for a neighbor x of y in A. This proves the claim.

Now let us use the above claim to prove this proposition. Let m be the Ramsey number $R(n_1 + n, 2n - 1, n + n_2, n + n_2 - 1)$ and $N = 2^{m+1}$. (The Ramsey number $R(n_1, n_2, \ldots, n_k)$ is the minimum c such that for every k-edge-coloring of the complete graph K_c, there is $i \in \{1, 2, \ldots, k\}$ such that it contains a complete subgraph on n_i vertices in which all edges are colored by i.)

Let S be an independent set of size N. Then $(S, \emptyset, \emptyset)$ is a regular triple. By the claim, there exists a regular triple (A, X_0, Y_0) with $|X_0| = |Y_0| = m$. Let $X_0 = \{x_1, x_2, \ldots, x_m\}$ and $Y_0 = \{y_1, y_2, \ldots, y_m\}$ be ordered according to the definition of a regular triple.

Let K_m be the complete graph on the vertex set $\{1, 2, \ldots, m\}$. Let us color the edges of K_m as follows: The color of an edge ij $(i < j)$ of K_m is defined as $(a, b) \in \{0, 1\}^2$ such that

$$\begin{aligned} a = 1 &\text{ if and only if } x_i \text{ is adjacent to } y_j, \\ b = 1 &\text{ if and only if } y_i \text{ is adjacent to } y_j. \end{aligned}$$

By Ramsey's Theorem, there exists a subset I of $\{1, 2, \ldots, m\}$ such that all edges in $K_m[I]$ have the same color (a, b) and

$$|I| = \begin{cases} n_1 + n & \text{if } (a, b) = (0, 0), \\ 2n - 1 & \text{if } (a, b) = (1, 0), \\ n + n_2 & \text{if } (a, b) = (0, 1), \\ n + n_2 - 1 & \text{if } (a, b) = (1, 1). \end{cases}$$

Let us define the following sets.

$$\begin{aligned} I_1 &= \{i \in I : x_i \text{ is adjacent to } y_i\}, \\ I_2 &= I \setminus I_1, \\ Z_1 &= \{x_i : i \in I_1\} \cup \{y_i : i \in I_1\}, \\ Z_2 &= \{x_i : i \in I_2\} \cup \{y_i : i \in I_2\}, \\ Z_3 &= \{x_i : i \in I_2,\ i > j \text{ for some } j \in I_2\} \\ &\qquad \cup \{y_i : i \in I_2,\ i < j \text{ for some } j \in I_2\}. \end{aligned}$$

If $(a,b) = (0,0)$ and $|I_1| \geq n_1$, then $G[Z_1]$ has an induced subgraph isomorphic to n_1K_2. If $(a,b) = (0,0)$ and $|I_2| \geq n+1$, then $G[Z_3]$ has an induced subgraph isomorphic to a half-graph of height n.

If $(a,b) = (1,0)$ and $|I_1| \geq n$, then $G[Z_1]$ has an induced subgraph isomorphic to a half-graph of height n. If $(a,b) = (1,0)$ and $|I_2| \geq n$, then $G[Z_2]$ has an induced subgraph isomorphic to $\overline{L(K_{2,n})}$.

If $(a,b) = (0,1)$ and $|I_1| \geq n$, then $G[Z_1]$ has an induced subgraph isomorphic to a thin spider with n legs. If $(a,b) = (0,1)$ and $|I_2| \geq n_2 + 1$, then $G[Z_3]$ has an induced subgraph isomorphic to a half split graph of height n_2.

If $(a,b) = (1,1)$ and $|I_1| \geq n_2$, then $G[Z_1]$ has an induced subgraph isomorphic to a half split graph of height n_2. If $(a,b) = (1,1)$ and $|I_2| \geq n$, then $G[Z_2]$ has an induced subgraph isomorphic to a thick spider with n legs. □

4. Making use of a big induced matching

Outcome (4) of Proposition 3.1 states that the graph contains a large induced matching. By Corollary 2.3, we only need to consider graphs that contain no long chain. In the next proposition, we deal with the case when a prime graph has a large induced matching and no long chain.

Proposition 4.1. *Let $t \geq 2$ and n, n' be positive integers. Let $h(n,n',2) = n$ and*

$$h(n,n',i) = (n-1)R(n,n,n,n,n,n,n,n',n',h(n,n',i-1)) + 1$$

for an integer $i > 2$. Let v be a vertex of a graph G and let M be an induced matching of G consisting of $h(n,n',t)$ edges not incident with v. If for each edge $e = xy$ in M, there is a chain of length at most t from $\{x,y\}$ to v, then G has an induced subgraph isomorphic to one of the following:

(1) $K_{1,n}^{(1)}$,

(2) the half-graph of height n,

(3) $\overline{L(K_{2,n})}$*,*
(4) a spider with n *legs, or*
(5) the half split graph of height n'*.*

Proof. We proceed by induction on t. If $t = 2$, then trivially $V(M) \cup \{v\}$ induces a subgraph of G isomorphic to $K_{1,n}^{(1)}$. So we may assume that $t > 2$.

We will find an induced matching M' consisting of $h(n, n', t-1)$ edges not incident with v such that G has a chain of length at most $t-1$ from each edge in M' to v. For an edge e in M, let C_e be a chain of length at most t from e to v.

If there are n chains of length 2 in $\{C_e : e \in M\}$, then we obtain an induced subgraph isomorphic to $K_{1,n}^{(1)}$. Thus, we may assume that there exists a subset M_1 of M such that the length of C_e is larger than 2 for all $e \in M_1$ and $|M_1| \geq |M| - n + 1$. We say that a vertex w is *mixed* on an edge e if w is not an end of e and w is adjacent to exactly one end of e. Note that the first and second vertices of C_e are the ends of e and the third vertex of C_e is mixed on e.

If a vertex w is the third vertex of at least n chains in $\{C_e : e \in M_1\}$, then the vertex w with those n edges in M_1 induces a subgraph isomorphic to $K_{1,n}^{(1)}$. Thus we may choose a subset M_2 of M_1 such that $|M_2| \geq \lceil \frac{1}{n-1}|M_1| \rceil$ and the third vertices of chains in $\{C_e : e \in M_2\}$ are distinct. Since M_2 is an induced matching, no third vertex of C_e is an end of some edge in M_2.

Let $m = |M_2|$. Let $e_1, e_2, \ldots, e_m$ be the edges in M_2. For each edge e_i, let z_i be the third vertex in C_{e_i} mixed on e_i. Let x_i, y_i be the ends of e_i such that y_i is adjacent to z_i.

Let us construct an edge-coloring of K_m on the vertex set $\{1, 2, \ldots, m\}$ as follows. For $1 \leq i < j \leq m$, we color the edge ij of K_m by one of the 10 colors $(a, b, c) \in \{0, 1\}^3 \cup \{(2, 2, 2), (3, 3, 3)\}$ depending on the adjacencies of the pairs (z_i, z_j), (z_i, y_j), (y_i, z_j). We set $(a, b, c) = (2, 2, 2)$ if z_i is mixed on e_j, and $(a, b, c) = (3, 3, 3)$ if z_i is not mixed on e_j and z_j is mixed on e_i. If z_i is not mixed on e_j and z_j is not mixed on e_i, then we assign colors $(a, b, c) \in \{0, 1\}^3$ as follows; we set $a = 1$ if z_i, z_j are adjacent, $b = 1$ if z_i and y_j are adjacent, and $c = 1$ if y_i, z_j are adjacent.

We apply Ramsey's theorem to K_m and obtain a subset I of $V(K_m)$ such that all edges in $K_m[I]$ have the same color (a, b, c) and

$$|I| = \begin{cases} h(n, n', t-1) & \text{if } (a, b, c) = (0, 0, 0), \\ n' & \text{if } (a, b, c) \in \{(1, 1, 0), (1, 0, 1)\}, \\ n & \text{otherwise.} \end{cases}$$

Let G_1 be the subgraph of G induced on x_i, z_i for all $i \in I$. Let G_2 be the subgraph of G induced on y_i, z_i for all $i \in I$.

If $(a,b,c) = (2,2,2)$ or $(3,3,3)$, then the subgraph of G induced on x_i, y_i for all $i \in I$ and one vertex z_j for the maximum or minimum j in I is isomorphic to $K^{(1)}_{1,n}$.

If $(a,b,c) = (1,0,0)$, then G_2 is a thin spider with n legs. If $(a,b,c) = (1,1,1)$, then G_1 is a thick spider with n legs. If $(a,b,c) = (1,1,0)$ or $(1,0,1)$, then G_2 is a half split graph of height n'. If $(a,b,c) = (0,1,0)$ or $(0,0,1)$, then G_2 is a half-graph of height n. If $(a,b,c) = (0,1,1)$, then G_1 is isomorphic to $\overline{L(K_{2,n})}$.

If $(a,b,c) = (0,0,0)$, then let $M' = \{y_i z_i : i \in I\}$. Clearly M' is an induced matching of size $h(n,n',t-1)$ and $C_{x_i y_i} \setminus \{x_i\}$ is a chain of length at most $t-1$ from $\{y_i, z_i\}$ to v for each $e \in M'$. Since C_e has at least 4 vertices for each $e \in M_1$, no edge of M' is incident with v. We deduce the conclusion by applying the induction hypothesis to M'. □

5. Prime graphs containing a large half split graph

Since the graphs in outcomes (1)–(3) of Proposition 3.1 are prime and we have just dealt with outcome (4), it remains to handle outcome (5), the situation when a prime graph contains a large half split graph as an induced subgraph. We do that in this section.

Proposition 5.1. *For every positive integer n, there exists $N = g(n)$ such that every prime graph having a half split graph of height at least N as an induced subgraph contains a chain of length $n+1$ or an induced subgraph isomorphic to one of $H'_{n,I}$, H^*_n, and $\overline{H^*_n}$.*

Proof. Let $N = 4^{n-2}(n+1) + 2(n-2) + 1$. Let G be a graph having H_N as an induced subgraph. Let $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N$ be the vertices of a half split subgraph H'_N such that $\{a_1, a_2, \ldots, a_N\}$ is independent and $\{b_1, b_2, \ldots, b_N\}$ is a clique and a_i is adjacent to b_j if and only if $i \geq j$.

Suppose that G has no chain of length $n+1$ and no induced subgraph isomorphic to $H'_{n,I}$, H^*_n, or $\overline{H^*_n}$. Let $u_0, u_1, u_2, \ldots, u_t$ be a shortest chain from $\{a_N, b_N\}$ to a_1 or b_1 where $u_0 = a_N$ and $u_1 = b_N$.

Let $N_1 = N - 2(n-2) = 4^{n-2}(n+1) + 1$. Since $t \leq n$, there exists $1 = i_1 < i_2 < i_3 < \ldots < i_{N_1} = N$ such that neither a_{i_j} nor b_{i_j} is in $\{u_2, u_3, \ldots, u_{t-1}\}$ for all $j \in \{1, 2, \ldots, N_1\}$. The subgraph of G induced on $\{a_{i_1}, a_{i_2}, \ldots, a_{i_{N_1}}, b_{i_1}, b_{i_2}, \ldots, b_{i_{N_1}}\}$ is isomorphic to H'_{N_1}.

Let $N_2 = \lceil N_1/4^{n-2} \rceil = n+2$. By the pigeonhole principle, there is a subsequence $p_1 = 1 < p_2 < p_3 < \ldots < p_{N_2} = N$ of $i_1, i_2, \ldots, i_{N_1}$ such that none of $u_2, u_3, u_4, \ldots, u_{t-1}$ is mixed on $A = \{a_{p_2}, a_{p_3}, \ldots, a_{p_{(N_2-1)}}\}$

or mixed on $B = \{b_{p_2}, b_{p_3}, \ldots, b_{p_{(N_2-1)}}\}$. Note that $|A| = |B| = n$. Again, the subgraph of G induced on $\{a_1, a_N, b_1, b_N\} \cup A \cup B$ is isomorphic to H'_{N_2}.

By the construction, both u_0 and u_1 are anti-complete to A and complete to B. Let i be the minimum such that u_i is complete to A or anti-complete to B. Then such i exists because $u_t \in \{a_1, b_1\}$ is either complete to A or anti-complete to B.

If u_i is complete to A and anti-complete to B, then the subgraph of G induced on $A \cup B \cup \{u_i\}$ is isomorphic to $H'_{n,I}$.

If u_i is complete to $A \cup B$, then let $p \in \{u_{i-1}, u_{i-2}\}$ be a vertex non-adjacent to u_i. By the hypothesis, p is complete to B and anti-complete to A. It follows that the subgraph of G induced on

$$(A \cup B \cup \{p, u_i\}) \setminus \{b_{p_2}\}$$

is isomorphic to $\overline{H^*_n}$.

If u_i is anti-complete to $A \cup B$, then let $q \in \{u_{i-1}, u_{i-2}\}$ be a vertex adjacent to u_i. By the hypothesis, u_i is complete to B and anti-complete to A and therefore the subgraph of G induced on

$$(A \cup B \cup \{q, u_i\}) \setminus \{a_{p_{N_2-1}}\}$$

is isomorphic to H^*_n. □

6. Proof of the main theorem

Now we are ready to put all these things together to prove Theorem 1.2.

Proof of Theorem 1.2. We may assume that G has no chain of length $n+1$ by Corollary 2.3. Let f, g, h be functions given by Propositions 3.1, 5.1, and 4.1. Let $m = f(n, h(n, g(n), n), g(n))$ and $N = R(m, m)$.

Let G be a prime graph with at least N vertices. By Ramsey's Theorem, G has an independent set or a clique of size m. Since $\overline{G}$ is also prime, we may assume that G has an independent set of size m.

We may assume that G has no half split graph of height $g(n)$ by Proposition 5.1.

By Proposition 3.1, one of the following holds:

(i) G has an induced subgraph isomorphic to a spider with n legs.
(ii) G has an induced subgraph isomorphic to $\overline{L(K_{2,n})}$.
(iii) G has an induced subgraph isomorphic to the half-graph of height n.
(iv) G has an induced matching of size $h(n, g(n), n)$.

If (i), (ii), or (iii) holds, then we are done.

If (iv) holds, then let M be an induced matching of size $h(n, g(n), n)$ and let v be a vertex incident with no edge in M. For each edge $e = xy$ of M, there is a chain from $\{x, y\}$ to v because G is prime by Proposition 2.1. Since we assumed that there is no chain of length $n+1$, such chains have length at most n. By Proposition 4.1, G contains an induced subgraph isomorphic to either $K_{1,n}^{(1)}$, a half-graph of height n, $\overline{L(K_{2,n})}$, or a spider with n legs. This completes the proof. □

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