UNAVOIDABLE VERTEX-MINORS IN LARGE PRIME GRAPHS

O-JOUNG KWON AND SANG-IL OUM

Abstract. A graph is *prime* (with respect to the split decomposition) if its vertex set does not admit a partition (A, B) (called a *split*) with $|A|$, $|B| \geqslant 2$ such that the set of edges joining A and B induces a complete bipartite graph.

We prove that for each n, there exists N such that every prime graph on at least N vertices contains a vertex-minor isomorphic to either a cycle of length n or a graph consisting of two disjoint cliques of size n joined by a matching.

1. Introduction

In this paper, all graphs are simple and undirected. We write P_n and C_n to denote a graph that is a path and a cycle on n vertices, respectively. We aim to find analogues of the following theorems.

- (Ramsey's theorem)
  For every n, there exists N such that every graph on at least N vertices contains an induced subgraph isomorphic to K_n or $\overline{K_n}$.
- (folklore; see Diestel's book [8, Proposition 9.4.1])
  For every n, there exists N such that every *connected* graph on at least N vertices contains an induced subgraph isomorphic to K_n, $K_{1,n}$, or P_n.
- (folklore; see Diestel's book [8, Proposition 9.4.2])
  For every n, there exists N such that every 2-*connected* graph on at least N vertices contains a topological minor isomorphic to C_n or $K_{2,n}$.
- (Oporowski, Oxley, and Thomas [15])

Date: March 24, 2014.

Key words and phrases. vertex-minor, split decomposition, blocking sequence, prime, generalized ladder.

Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2011-0011653).

For every n, there exists N such that every 3-*connected* graph on at least N vertices contains a minor isomorphic to the wheel graph W_n on n vertices or $K_{3,n}$.

- (Ding, Chen [9])

  For every integer n, there exists N such that every *connected and co-connected* graph on at least N vertices contains an induced subgraph isomorphic to P_n, $K^s_{1,n}$ (the graph obtained from $K_{1,n}$ by subdividing one edge once), $K_{2,n}\backslash e$, or $K_{2,n}/e\backslash f\backslash g$ where $\{f, g\}$ is a matching in $K_{2,n}/e$. A graph is *co-connected* if its complement graph is connected.
- (Chun, Ding, Oporowski, and Vertigan [6])

  For every integer $n \geqslant 5$, there exists N such that every internally 4-connected graph on at least N vertices contains a parallel minor isomorphic to K_n, $K'_{4,n}$ ($K_{4,n}$ with a complete graph on the vertices of degree n), TF_n (the n-partition triple fan with a complete graph on the vertices of degree n), D_n (the n-spoke double wheel), D'_n (the n-spoke double wheel with axle), M_n (the $(2n+1)$-rung Mobius zigzag ladder), or Z_n (the $(2n)$-rung zigzag ladder).

These theorems commonly state that every sufficiently large graph having certain connectivity contains at least one graph in the list of *unavoidable* graphs by certain graph containment relation. Moreover in each theorem, the list of unavoidable graphs is *optimal* in the sense that each unavoidable graph in the list has the required connectivity, can be made arbitrary large, and does not contain other unavoidable graphs in the list.

In this paper, we discuss *prime* graphs as a connectivity requirement. A *split* of a graph G is a partition (A, B) of the vertex set $V(G)$ having subsets $A_0 \subseteq A$, $B_0 \subseteq B$ such that $|A|, |B| \geqslant 2$ and a vertex $a \in A$ is adjacent to a vertex $b \in B$ if and only if $a \in A_0$ and $b \in B_0$. This concept was first studied by Cunningham [7] in his research on split decompositions. We say that a graph is *prime* if it has no splits. Sometimes we say a graph is *prime with respect to split decomposition* to distinguish with another notion of primeness with respect to modular decomposition.

Prime graphs play important role in the study of circle graphs (intersection graphs of chords in a circle) and their recognition algorithms. Bouchet [2], Naji [14], and Gabor, Hsu, and Supowit [11] independently showed that prime circle graphs have a unique chord diagram. This is comparable to the fact that 3-connected planar graphs have a unique planar embedding.

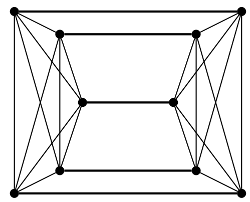


FIGURE 1. $K_5 \boxminus K_5$.

The graph containment relation we will mainly discuss is called a *vertex-minor*. A graph H is a *vertex-minor* of a graph G if there exist a sequence $v_1, v_2, \ldots, v_n$ of (not necessarily distinct) vertices and a subset $X \subseteq V(G)$ such that $H = G * v_1 * v_2 \cdots * v_n \setminus X$, where $G * v$ is an operation called *local complementation*, to take the complement graph only in the neighborhood of v. The detailed description will be given in Section 2.1. Vertex-minors are important in circle graphs; for instance, Bouchet [5] proved that a graph is a circle graph if and only if it has no vertex-minor isomorphic to one of three particular graphs.

Prime graphs have been studied with respect to vertex-minors, perhaps because local complementation preserves prime graphs, shown by Bouchet [2]. In addition, he showed the following.

Theorem 1.1 (Bouchet [2])**.** *Every prime graph on at least* 5 *vertices must contain a vertex-minor isomorphic to* C_5*.*

Here is the main theorem of this paper.

Theorem 7.1. *For every n, there is N such that every prime graph on at least N vertices has a vertex-minor isomorphic to C_n or $K_n \boxminus K_n$.*

The graph $K_n \boxminus K_n$ is a graph obtained by joining two copies of K_n by a matching of size n, see Figure 1. This notation will be explained in Section 2.4. In addition, we show that this list of unavoidable vertex-minors in Theorem 7.1 is optimal, which will be discussed in Section 8. We will heavily use Ramsey's theorem iteratively and so our bound N is astronomical in terms of n.

The proof is splitted into two parts.

(1) We first prove that for each n, there exists N such that every prime graph having an induced path of length N contains a vertex-minor isomorphic to C_n. (In fact, we prove that $N = \lceil 6.75n^7 \rceil$.)
(2) Secondly, we prove that for each n, there exists N such that every prime graph on at least N vertices contains a vertex-minor isomorphic to P_n or $K_n \boxminus K_n$.

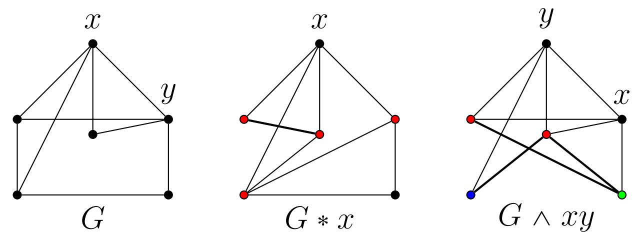


FIGURE 2. Local complementation and pivot.

To prove (1), we actually prove first that every sufficiently large generalized ladder, a certain type of outerplanar graphs, contains C_n as a vertex-minor. This will be shown in Section 4. Then, we use the technique of blocking sequences developed by Geelen [13] to construct a large generalized ladder in a prime graph having a sufficiently long induced path, shown in Section 6. Blocking sequences will be discussed and developed in Section 5. The second part (2) is discussed in Section 7, where we iteratively use Ramsey's theorem to find a bigger configuration called a broom inside a graph. In Section 3, we give similar theorems of this type on vertex-minors with respect to less restrictive connectivity requirements.

2. PRELIMINARIES

For $X \subseteq V(G)$, let $\delta_G(X)$ be the set of edges having one end in X and another end in $V(G)\setminus X$. Let $N_G(x)$ be the set of the neighbors of a vertex x in G. For $X \subseteq V(G)$, let $G[X]$ be the induced subgraph of G on the vertex set X. For two disjoint subsets S, T of $V(G)$, let $G[S,T] = G[S \cup T]\setminus(E(G[S]) \cup E(G[T]))$. Clearly, $G[S,T]$ is a bipartite graph with the bipartition (S,T).

2.1. **Vertex-minors.** The *local complementation* of a graph G at a vertex v is an operation to replace the subgraph of G induced by the neighborhood of v by its complement graph. In other words, to apply local complementation at v for every pair x, y of neighbors of v, we flip the pair x, y, where *flipping* means that we delete the edge if it exists and add it otherwise. We write $G*v$ to denote the graph obtained from G by applying local complementation of G at v. Two graphs are *locally equivalent* if one is obtained from another by applying a sequence of local complementations. A graph H is a *vertex-minor* of G if H is an induced subgraph of a graph locally equivalent to G.

For an edge xy of a graph G, a graph obtained by *pivoting* an edge xy of G is defined as $G \wedge xy = G * x * y * x$. Here is a direct way to see

$G \wedge xy$; there are 3 kinds of neighbors of x or y; some are adjacent to both, some are adjacent to only x, others are adjacent to only y. We flip the adjacency between all pairs of neighbors of x or y of distinct kinds and then swap the two vertices x and y. Two graphs are *pivot-equivalent* if one is obtained from another by a sequence of pivots. Thus, pivot-equivalent graphs are locally equivalent. See Figure 2 for an example of these operations.

The following lemma by Bouchet provides a key tool to investigate vertex-minors. His proof is based on isotropic systems, which are some linear algebraic objects corresponding to the equivalence classes of graphs with respect to local equivalence, introduced by Bouchet [1]. A direct proof is given by Geelen and Oum [12].

Lemma 2.1 (Bouchet [3]; see Geelen and Oum [12])**.** *Let H be a vertex-minor of G and let $v \in V(G)\backslash V(H)$. Then H is a vertex-minor of $G\backslash v$, $G * v\backslash v$, or $G \wedge vw\backslash v$ for a neighbor w of v.*

The choice of a neighbor w in Lemma 2.1 does not matter, because if x is adjacent to y and z, then $G \wedge xy = (G \wedge xz) \wedge yz$ (see [16]).

2.2. **Cut-rank function.** Let $A(G)$ be the adjacency matrix of G over the binary field. For an $X \times Y$ matrix A, if $X' \subseteq X$ and $Y' \subseteq Y$, then we write $A[X', Y']$ to denote the submatrix of A obtained by taking rows in X' and columns in Y'.

We define $\rho^*_G(X, Y) = \operatorname{rank} A(G)[X, Y]$. This function satisfies the following submodular inequality (see Oum and Seymour [18]):

Lemma 2.2 (See Oum and Seymour [18])**.** *For all $A, B, A', B' \subseteq V(G)$,*

$$\rho^*_G(A, B) + \rho^*_G(A', B') \geqslant \rho^*_G(A \cap A', B \cup B') + \rho^*_G(A \cup A', B \cap B').$$

The *cut-rank* function ρ_G of a graph G is defined as

$$\rho_G(X) = \rho^*_G(X, V(G)\backslash X) = \operatorname{rank} A(G)[X, V(G)\backslash X].$$

By Lemma 2.2, we have the submodular inequality:

$$\rho_G(A) + \rho_G(B) \geqslant \rho_G(A \cap B) + \rho_G(A \cup B)$$

for all $A, B \subseteq V(G)$.

The cut-rank function is invariant under taking local complementation, which makes it useful for us.

Lemma 2.3 (Bouchet [4]; See Oum [16])**.** *If G and H are locally equivalent, then $\rho_G(X) = \rho_H(X)$ for all $X \subseteq V(G)$.*

Lemma 2.4 (Oum [16, Lemma 4.4])**.** *Let G be a graph and $v \in V(G)$. Suppose that (X_1, X_2), (Y_1, Y_2) are partitions of $V(G)\backslash\{v\}$. Then we have*

$$\rho_{G\backslash v}(X_1) + \rho_{G*v\backslash v}(Y_1) \geqslant \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.$$

Similarly if w is a neighbor of v, then

$$\rho_{G\backslash v}(X_1) + \rho_{G\wedge vw\backslash v}(Y_1) \geqslant \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.$$

Lemma 2.4 is equivalent to the following lemma, which we will use in the proof of Proposition 5.3.

Lemma 2.5. *Let G be a graph and $v \in V(G)$. Suppose that X_1, X_2, Y_1, Y_2 are subsets of $V(G)\backslash\{v\}$ such that $X_1 \cup X_2 = Y_1 \cup Y_2$ and $X_1 \cap X_2 = Y_1 \cap Y_2 = \varnothing$. Then*

$$\begin{aligned}&\rho^*_G(X_1, X_2) + \rho^*_{G*v}(Y_1, Y_2)\\ &\quad\geqslant \rho^*_G(X_1 \cap Y_1, X_2 \cup Y_2 \cup \{v\}) + \rho^*_G(X_1 \cup Y_1 \cup \{v\}, X_2 \cap Y_2) - 1.\end{aligned}$$

Similarly if $w \in X_1 \cup X_2$ is a neighbor of v, then

$$\begin{aligned}&\rho^*_G(X_1, X_2) + \rho^*_{G\wedge vw}(Y_1, Y_2)\\ &\quad\geqslant \rho^*_G(X_1 \cap Y_1, X_2 \cup Y_2 \cup \{v\}) + \rho^*_G(X_1 \cup Y_1 \cup \{v\}, X_2 \cap Y_2) - 1.\end{aligned}$$

Proof. Apply Lemma 2.4 with $G' = G[X_1 \cup X_2 \cup \{v\}]$. □

2.3. **Prime graphs.** For a graph G, a partition (A, B) of $V(G)$ is called a *split* if $|A|, |B| \geqslant 2$ and there exist $A' \subseteq A$ and $B' \subseteq B$ such that $x \in A$ is adjacent to $y \in B$ if and only if $x \in A'$ and $y \in B'$. A graph is *prime* (with respect to the split decomposition) if it has no splits. These concepts were introduced by Cunningham [7].

Alternatively, a split can be understood with the *cut-rank* function ρ_G. A partition (A, B) of $V(G)$ is a split if and only if $|A|, |B| \geqslant 2$ and yet $\rho_G(A) \leqslant 1$.

The following lemma is natural.

Lemma 2.6. *If a prime graph H on at least 5 vertices is a vertex-minor of a graph G, then G has a prime induced subgraph G_0 such that G_0 has a vertex-minor isomorphic to H.*

Proof. We may assume that G is connected. It is enough to prove the following claim: if G has a split (A, B), then there exists a vertex v such that H is isomorphic to a vertex-minor of $G\backslash v$. Let G' be a graph locally equivalent to G such that H is an induced subgraph of G'. We have $\rho_H(V(H) \cap A) = \rho^*_{G'}(V(H) \cap A, V(H) \cap B) \leqslant \rho^*_{G'}(A, B) \leqslant 1$ and therefore $|V(H) \cap A| \leqslant 1$ or $|V(H) \cap B| \leqslant 1$ because H is prime. By symmetry, let us assume $|V(H) \cap B| \leqslant 1$. Let us choose $x \in B$ such that x has a neighbor in A and $x \in V(H)$ if $V(H) \cap B$ is nonempty.

Let H' be a vertex-minor of G on $A \cup \{x\}$ such that H is isomorphic to a vertex-minor of H'. Then $H' = G * v_1 * v_2 \cdots * v_n \backslash (B \backslash \{x\})$ for some sequence $v_1, v_2, \ldots, v_n$ of vertices. We may choose H' and n so that n is minimized.

Suppose $n > 0$. Then $v_n \in B \backslash \{x\}$. Let $H_0 = G * v_1 * v_2 \cdots * v_{n-1} \backslash (B \backslash \{x, v_n\})$. Since $(A, \{x, v_n\})$ is a split of H_0, one of the following holds.

(i) The two vertices v_n and x have the same set of neighbors in A.
(ii) The vertex v_n has no neighbors in A.
(iii) The vertex x has no neighbors in A.

If we have the case (i), then $(H_0 \backslash v_n) * x = H'$ and therefore H is isomorphic to a vertex-minor of $H_0 \backslash v_n$, contradicting our assumption that H is chosen to minimize n. If we have the case (ii), then $H_0 \backslash v_n = H'$, contradicting the assumption too. Finally if we have the case (iii), then x is adjacent to v_n in G because G is connected. Then $H_0 * v_n \backslash v_n$ is isomorphic to $H_0 * v_n \backslash x$. Then $H_0 \backslash x$ has a vertex-minor isomorphic to H, contradicting our assumption that n is minimized. □

2.4. **Constructions of graphs.** For two graphs G and H on the same set of n vertices, we would like to introduce operations to construct graphs on $2n$ vertices by making the disjoint union of them and adding some edges between two graphs. Roughly speaking, $G \boxminus H$ will add a perfect matching, $G \boxtimes H$ will add the complement of a perfect matching, and $G \boxslash H$ will add a bipartite chain graph. Formally, for two graphs G and H on $\{v_1, v_2, \ldots, v_n\}$, let $G \boxminus H$, $G \boxtimes H$, $G \boxslash H$ be graphs on $\{v_1^1, v_2^1, \ldots, v_n^1, v_1^2, v_2^2, \ldots, v_n^2\}$ such that for all $i, j \in \{1, 2, \ldots, n\}$,

(i) $v_i^1 v_j^1 \in E(G \boxminus H)$ if and only if $v_i v_j \in E(G)$,
(ii) $v_i^2 v_j^2 \in E(G \boxminus H)$ if and only if $v_i v_j \in E(H)$,
(iii) $v_i^1 v_j^2 \in E(G \boxminus H)$ if and only if $i = j$,
(iv) $v_i^1 v_j^1 \in E(G \boxtimes H)$ if and only if $v_i v_j \in E(G)$,
(v) $v_i^2 v_j^2 \in E(G \boxtimes H)$ if and only if $v_i v_j \in E(H)$,
(vi) $v_i^1 v_j^2 \in E(G \boxtimes H)$ if and only if $i \neq j$,
(vii) $v_i^1 v_j^1 \in E(G \boxslash H)$ if and only if $v_i v_j \in E(G)$,
(viii) $v_i^2 v_j^2 \in E(G \boxslash H)$ if and only if $v_i v_j \in E(H)$,
(ix) $v_i^1 v_j^2 \in E(G \boxslash H)$ if and only if $i \geqslant j$.

See Figure 3 for $K_5 \boxminus \overline{K_5}$, $K_5 \boxtimes \overline{K_5}$, and $K_5 \boxslash \overline{K_5}$.

We will use the following lemmas.

Lemma 2.7. *Let $n \geqslant 3$ be an integer.*

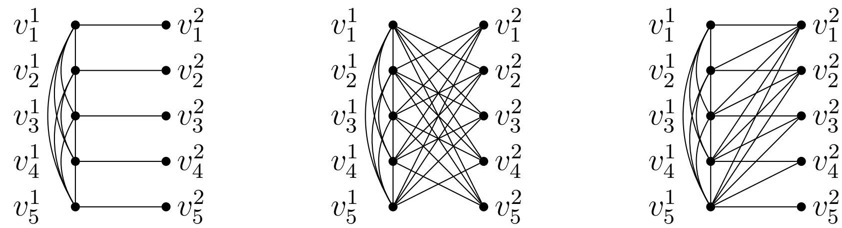


FIGURE 3. $K_5 \boxminus \overline{K_5}$, $K_5 \boxtimes \overline{K_5}$, and $K_5 \boxslash \overline{K_5}$.

(1) $K_n \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $K_{n-1} \boxminus K_{n-1}$.
(2) $\overline{K_n} \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $K_{n-2} \boxminus K_{n-2}$.

Proof. (1) Let $V(K_n) = V(\overline{K_n}) = \{v_i : 1 \leqslant i \leqslant n\}$. The graph $(K_n \boxtimes \overline{K_n}) * v_1^1 * v_1^2 \backslash v_1^1 \backslash v_1^2$ is isomorphic to $K_{n-1} \boxminus K_{n-1}$.

(2) Let $V(\overline{K_n}) = \{v_1, v_2, \ldots, v_n\}$. The graph $(\overline{K_n} \boxtimes \overline{K_n}) * v_1^1 \backslash v_1^1 \backslash v_1^2$ is isomorphic to $\overline{K_{n-1}} \boxminus K_{n-1}$. By (1), $\overline{K_n} \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $K_{n-2} \boxminus K_{n-2}$. □

Lemma 2.8. *Let n be a positive integer.*

(1) The graph $\overline{K_n} \boxslash \overline{K_n}$ is pivot-equivalent to P_{2n}.
(2) The graph $K_n \boxslash \overline{K_n}$ is locally equivalent to P_{2n}.

Proof. (1) Let $P = p_1 p_2 \ldots p_{2n}$. We can check that $\overline{K_n} \boxslash \overline{K_n}$ can be obtained from P by pivoting $p_i p_{i+1}$ for all $i = 1, 3, \ldots, 2n-1$.

(2) Let $V(K_n) = V(\overline{K_n}) = \{v_1, v_2, \ldots, v_n\}$. Since $(K_n \boxslash \overline{K_n}) * v_1^2$ is isomorphic to $\overline{K_n} \boxslash \overline{K_n}$, the result follows from (1). □

2.5. **Ramsey numbers.** A *clique* is a set of pairwise adjacent vertices. A *stable set* or an *independent set* is a set of pairwise non-adjacent vertices.

We write $R(n_1, n_2, \ldots, n_k)$ to denote the minimum number N such that in every k coloring of the edges of K_N, there exist i and a clique of size n_i whose edges are all colored with the i-th color. Such a number exists by Ramsey's theorem [19].

3. Unavoidable vertex-minors in large graphs

We present three simple statements on unavoidable vertex-minors. These are optimal as discussed in Section 1.

Theorem 3.1. *(1) For every n, there exists N such that every graph on at least N vertices has a vertex-minor isomorphic to $\overline{K_n}$.*

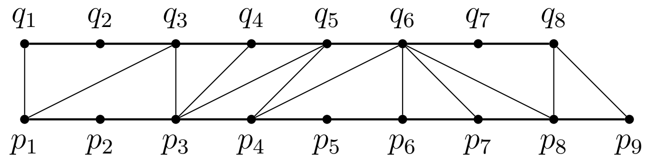


FIGURE 4. An example of a generalized ladder.

(2) *For every n, there exists N such that every connected graph having at least N vertices has a vertex-minor isomorphic to K_n.*
(3) *For every n, there exists N such that every graph having at least N edges has a vertex-minor isomorphic to K_n or $\overline{K_n} \boxminus \overline{K_n}$.*

Proof. (1) If a graph has no $\overline{K_n}$ as a vertex-minor, then it has no vertex-minor isomorphic to K_{n+1}. So we can take $N = R(n, n+1)$.

(2) Let us assume that G has no vertex-minor isomorphic to K_n. Then the maximum degree of G is less than $\Delta = R(n-1, n-1)$ by Ramsey theorem. If $|V(G)|$ is big enough, then it contains an induced path P of length $2n-3$ because the maximum degree is bounded. By Lemma 2.8, P_{2n-2} has a vertex-minor isomorphic to $K_{1,n-1}$, that is locally equivalent to K_n.

(3) Let G be a graph having no vertex-minor isomorphic to K_n or $\overline{K_n} \boxminus \overline{K_n}$. Each component of G has bounded number of vertices, say M, by (2). Since $\overline{K_n} \boxminus \overline{K_n}$ is not a vertex-minor of G, G has less than n non-trivial components. (A component is trivial if it has no edges.) So G has at most $\binom{M}{2}(n-1)$ edges. □

4. Obtaining a long cycle in a huge generalized ladder

A *generalized ladder* is a graph G with two vertex-disjoint paths $P = p_1p_2 \dots p_a$, $Q = q_1q_2 \dots q_b$ ($a, b \geqslant 1$) with additional edges, called *chords*, each joining a vertex of P with a vertex of Q such that $V(P) \cup V(Q) = V(G)$, p_1 is adjacent to q_1, p_a is adjacent to q_b, and no two chords cross. Two chords p_iq_j and $p_{i'}q_{j'}$ ($i < i'$) *cross* if and only if $j > j'$. We remark that a generalized ladder is a outerplanar graph whose weak dual is a path. We call p_1q_1 the *first chord* and p_aq_b the *last chord* of G. Since no two chords cross, p_1 or q_1 has degree at most 2. Similarly, p_a or q_b has degree at most 2. See Figure 4 for an example.

We will prove the following proposition.

Proposition 4.1. *Let $n \geqslant 2$. Every generalized ladder having at least $4608n^5$ vertices has a cycle of length $4n+3$ as a vertex-minor.*

4.1. **Lemmas on a fan.** Let F_n be a graph on n vertices with a specified vertex c, called the center, such that $F_n \backslash c$ is a path on $n-1$ vertices and c is adjacent to all other vertices. We call F_n a *fan* on n vertices.

Lemma 4.2. *A fan F_{3n} has a vertex-minor isomorphic to a cycle of length $2n+1$.*

Proof. Let c be the center of F_{3n}. Let $v_1, v_2, \ldots, v_{3n-1}$ be the non-center vertices in F_{3n} forming a path. Let $G = F_{3n} * v_3 * v_6 * v_9 \cdots * v_{3n-3}$. Clearly c is adjacent to v_i in G if and only if $i \in \{1, 3n-1\}$ or $i \equiv 0 \pmod 3$ and furthermore v_{3i-1} is adjacent to v_{3i+1} in G for all i. Let $H = G \backslash \{v_3, v_6, \ldots, v_{3n-3}\}$. Then H is a cycle of length $3n-(n-1)$. □

Lemma 4.3. *Let $n \geqslant 2$. Let G be a graph with a vertex c such that $G \backslash c$ is isomorphic to an induced path P whose both ends are adjacent to c. If $|V(G)| \geqslant 6(n-1)^2 - 3$, then G has a vertex-minor isomorphic to a cycle of length $2n+1$.*

Proof. We may assume that $n \geqslant 3$. Let $P = v_1 v_2 \ldots v_k$ with $k \geqslant 6$. We may assume that v_2 is adjacent to c because otherwise we replace G with $G * v_1$. Similarly we may assume that v_{k-1} is adjacent to c. We may also assume v_3 is adjacent to c because otherwise we replace G with $G \wedge v_1 v_2$. Similarly we may assume that v_{k-2} is adjacent to c.

If c is adjacent to at least $3n-1$ vertices on P, then G has a vertex-minor isomorphic to F_{3n}. So by Lemma 4.2, G has a vertex-minor isomorphic to a cycle of length $2n+1$. Thus we may assume that the number of neighbors of c is at most $3n-2$. The neighbors of c gives a partition of P into at most $3n-3$ subpaths. We already have 4 subpaths at both ends having length 1. Since

$$|E(P)| \geqslant 6(n-1)^2 - 3 - 2 > (2n-2)((3n-3)-4) + 4,$$

there exists a subpath P' of P having length at least $2n-1$ such that no internal vertex of P' is adjacent to c and the ends of P' are adjacent to c. This together with c gives an induced cycle of length at least $2n+1$. □

4.2. **Generalized ladders of maximum degree at most 3.**

Lemma 4.4. *Let G be a generalized ladder of maximum degree 3. If G has at least $6n$ vertices of degree 3, then G has a cycle of length $4n+3$ as a vertex-minor.*

Proof. We proceed by induction on $|V(G)|$. Let P, Q be two defining paths of G. We may assume that all internal vertices of P or Q has degree 3, because if P or Q has an internal vertex v of degree 2, then we apply the induction hypothesis to $G * v \backslash v$. Since p_1 or q_1 has degree

2, we may assume that p_1 has degree 2 by symmetry. We may assume that q_1 has degree 3 because otherwise we can apply the induction hypothesis to $G * q_1 \backslash q_1$. Consequently q_1 is adjacent to p_2 and thus for each internal vertex q_i of Q, q_i is adjacent to p_{i+1} and each internal vertex p_{i+1} of P is adjacent to q_i. Thus either $a = b$ and p_a has degree 3 or $a = b + 1$ and p_a has degree 2. But if $a = b + 1$ and p_a has degree 2, then we can apply the induction hypothesis to $G * p_a \backslash p_a$. Thus we may assume that $a = b$ and p_a has degree 3. Since G has at least $6n$ vertices of degree 3, $a > 3n$ and $b > 3n$. If $a = b > 3n + 1$, then we can apply the induction hypothesis to $G \backslash q_b$. Thus we may assume that $a = b = 3n + 1$ and p_a has degree 3 and q_b has degree 2. Note that p_i is adjacent to q_{i-1} for all $i = 2, \ldots, 3n + 1$. Then $G * p_1 \wedge p_4 q_3 \wedge p_7 q_6 \cdots \wedge p_{3n+1} q_{3n} \backslash \{p_4, p_7, \ldots, p_{3n-2}, q_3, q_6, \ldots, q_{3n-3}, q_{3n+1}\}$ is isomorphic to a cycle of length $4n + 3$. □

Lemma 4.5. *Let G be a generalized ladder of maximum degree 3. If $|V(G)| \geqslant 12n^2$, then G has a cycle of length $4n + 3$ as a vertex-minor.*

Proof. Let P, Q be two defining paths of G. We may assume $a > 1$ and $b > 1$ because otherwise G has an induced cycle of length at least $6n^2 + 1 \geqslant 4n + 3$.

Let $p_x q_y$ be the unique chord other than $p_1 q_1$ with minimum $x + y$. We claim that we may assume $(x-1)+(y-1) \leqslant 2$. Suppose not. Then $p_x q_y$, $p_1 q_1$ and subpaths of P and Q form a cycle of length $x + y \geqslant 5$ and $p_1, p_2, \ldots, p_{x-1}, q_1, q_2, \ldots, q_{y-1}$ have degree 2. By moving the first few vertices of P to Q or Q to P, we may assume that $x \geqslant 3$ and $y \geqslant 2$. Then we may replace G with $G * p_1$. This proves the claim.

Thus the induced cycle containing $p_1 q_1$ has at most 2 edges from $E(P) \cup E(Q)$. Similarly we may assume that the induced cycle containing $p_a q_b$ has at most 2 edges from $E(P) \cup E(Q)$.

If G has at least $6n$ vertices of degree 3, then by Lemma 4.4, we obtain a desired vertex-minor. So we may assume that G has at most $6n - 1$ vertices of degree 3. Thus G has at most $3n - 1$ chords other than $p_1 q_1$ and $p_a q_b$. These chords give at most $3n$ induced cycles of G where each edge in $E(P) \cup E(Q)$ appears in exactly one of them. If every such induced cycle has length at most $4n + 2$, then

$$|E(P) \cup E(Q)| \leqslant (3n - 2)(4n) + 4 = 12n^2 - 8n + 4 < 12n^2 - 2.$$

Since $|V(G)| \geqslant 12n^2$, we have $|E(P) \cup E(Q)| \geqslant 12n^2 - 2$. This leads to a contradiction. □

4.3. **Generalized ladders of maximum degree 4.**

Lemma 4.6. *Let G be a generalized ladder of maximum degree at most 4. Let α be the number of vertices of G having degree 3 or 4. Then G has a vertex-minor H that is a generalized ladder of maximum degree at most 3 such that $|V(H)| \geqslant \alpha/4$.*

Proof. Let $P = p_1p_2 \dots p_a$, $Q = q_1q_2 \dots q_b$ be the paths defining a generalized ladder G. Let $X_{i,j} = \{p_1, p_2, \dots, p_i, q_1, q_2, \dots, q_j\}$. We may assume $\alpha > 8$.

If $a = 1$, then p_1 has at least $\alpha - 1$ neighbors but the maximum degree is 4 and therefore $\alpha \leqslant 5$, contradicting our assumption. Thus $a > 1$. Similarly $b > 1$.

We may also assume that no internal vertex of P or Q has degree 2, because otherwise we can apply local complementation and remove it.

Let $\alpha_{i,j}(G)$ be the number of vertices in $V(G) \backslash X_{i,j}$ having degree 3 or 4. We will prove the following.

Claim 1. *Suppose that there exist $1 \leqslant i < a$ and $1 \leqslant j < b$ such that $\delta_G(X_{i,j})$ has exactly two edges and every vertex in $X_{i,j}$ has degree 2 or 3 in G. Then G has a vertex-minor H that is a generalized ladder of maximum degree at most 3 such that $|V(H)| \geqslant |X_{i,j}| + \alpha_{i,j}(G)/4$.*

Before proving Claim 1, let us see why this claim implies our lemma. First we would like to see why there exist i and j such that $\delta_G(X_{i,j})$ has exactly two edges. If p_1 has degree bigger than 2, then p_1 is adjacent to q_2 and so $G * q_1 = G \backslash p_1q_2$. Thus we may assume that both p_1 and q_1 have degree 2. Keep in mind that the number of vertices of degree 3 or 4 in $X_{1,1}$ may be decreased by 1 by replacing G with $G * q_1$ and so $\alpha_{1,1}(G) \geqslant \alpha - 2$.

By applying Claim 1 with $i = j = 1$, we obtain a generalized ladder H of maximum degree at most 3 as a vertex-minor such that $|V(H)| \geqslant 2 + (\alpha - 2)/4 \geqslant \alpha/4$. This completes the proof of the lemma, assuming Claim 1.

We now prove Claim 1 by induction on $|V(G)| - |X_{i,j}(G)|$. We may assume that every vertex in $V(G) \backslash (X_{i,j} \cup \{p_a, q_b\})$ has degree 3 or 4 because otherwise we can apply local complementation and delete it while keeping $\alpha_{i,j}$. Then p_{i+1} is obviously adjacent to q_{j+1}.

We may assume that $i < a - 1$ because otherwise G is a generalized ladder of maximum degree 3 if p_a has degree 3 and $G \backslash q_b$ is a generalized ladder of maximum degree 3 otherwise. Similarly we may assume $j < b - 1$. Either p_{i+1} or q_{j+1} has degree 4, because otherwise $\delta_G(X_{i+1,j+1})$ has exactly two edges. By symmetry, we may assume that p_{i+1} has degree 3 and q_{j+1} has degree 4 and therefore q_{j+1} is adjacent to p_{i+2}.

If $\alpha_{i,j}(G) \leqslant 12$, then $H = G[X_{i+2,j+1}]$ is a generalized ladder of maximum degree at most 3. Thus we may assume that $\alpha_{i,j}(G) > 12$. If $b - j \leqslant 4$, then $a - i \leqslant 8$ because each vertex in $q_{j+1}, q_{j+2}, \ldots, q_b$ has degree at most 4 and each vertex in $p_{i+1}, p_{i+2}, \ldots, p_{a-1}$ has degree at least 3. This contradicts our assumption that $\alpha_{i,j}(G) > 12$. So we may assume that $b - j \geqslant 5$ and similarly $a - i \geqslant 5$.

Let R be the component of $G \backslash (E(P) \cup E(Q))$ containing p_{i+1}. Because of the degree condition, R is a path. We now consider six cases, see Figure 5.

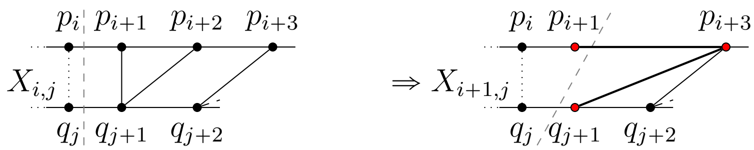
(a) If R has length 2 and p_{i+3} has degree 3 in G, then $G' = G * p_{i+2} \backslash p_{i+2} = (G \backslash p_{i+2} + p_{i+1}p_{i+3} + q_{j+1}p_{i+3}) \backslash p_{i+1}q_{j+1}$ is a generalized ladder of maximum degree at most 4. Every vertex in G' not in $X_{i,j}$ has degree at most 4. Furthermore p_{i+1} has degree 2 in G'. Thus, $\delta_{G'}(X_{i+1,j})$ has exactly 2 edges. Then $|X_{i+1,j}| + \alpha_{i+1,j}(G')/4 \geqslant (|X_{i,j}| + 1) + (\alpha_{i,j}(G) - 2)/4 \geqslant |X_{i,j}| + \alpha_{i,j}(G)/4$. By the induction hypothesis, we find a desired vertex-minor H in G'.

(b) If R has length 2 and p_{i+3} has degree 4 in G, then the vertex q_{j+2} has degree 3. Then $G' = G * p_{i+2} * q_{j+2} \backslash p_{i+2} \backslash q_{j+2}$ is a generalized ladder of maximum degree at most 4. Then $\delta_{G'}(X_{i+1,j+1})$ has exactly two edges and $\alpha_{i+1,j+1}(G') \geqslant \alpha_{i,j}(G) - 6$. Again, $|X_{i+1,j+1}| + \alpha_{i+1,j+1}(G')/4 \geqslant |X_{i,j}| + 2 + (\alpha_{i,j}(G) - 6)/4 \geqslant |X_{i,j}| + \alpha_{i,j}(G)/4$ and therefore we are done.

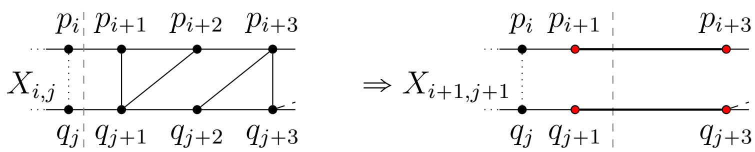
(c) If R has length 3 and q_{j+3} has degree 3 in G, then $G' = G * q_{j+2} \backslash q_{j+2}$ is a generalized ladder of maximum degree at most 4. Then $\delta_{G'}(X_{i+1,j+1})$ has exactly two edges and $\alpha_{i+1,j+1}(G') \geqslant \alpha_{i,j}(G) - 3$. We deduce that $|X_{i+1,j+1}| + \alpha_{i+1,j+1}(G')/4 \geqslant |X_{i,j}| + 2 + (\alpha_{i,j}(G) - 3)/4 \geqslant |X_{i,j}| + \alpha_{i,j}(G)/4$.

(d) If R has length 3 and q_{j+3} has degree 4 in G, then p_{i+3} has degree 3 and $G' = G * q_{j+2} * p_{i+3} \backslash q_{j+2} \backslash p_{i+3}$ is a generalized ladder of maximum degree at most 4. Then $\delta_{G'}(X_{i+2,j+1})$ has exactly two edges and $\alpha_{i+2,j+1}(G') \geqslant \alpha_{i,j}(G) - 7$. We deduce that $|X_{i+2,j+1}| + \alpha_{i+2,j+1}(G')/4 \geqslant |X_{i,j}| + 3 + (\alpha_{i,j}(G) - 7)/4 \geqslant |X_{i,j}| + \alpha_{i,j}(G)/4$. By the induction hypothesis, G' has a desired vertex-minor and so does G.

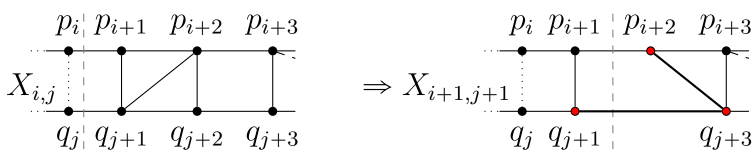
(e) If R has length 4, then $G' = G \wedge p_{i+2}q_{j+2} * p_{i+3} \backslash p_{i+2} \backslash p_{i+3} \backslash q_{j+2}$ is a generalized ladder of maximum degree at most 4. Then $\delta_{G'}(X_{i+1,j+1})$ has exactly two edges and $\alpha_{i+1,j+1}(G') \geqslant \alpha_{i,j}(G) - 7$ and therefore $|X_{i+1,j+1}| + \alpha_{i+1,j+1}(G')/4 \geqslant |X_{i,j}| + 2 + (\alpha_{i,j}(G) - 7)/4 \geqslant |X_{i,j}| + \alpha_{i,j}(G)/4$. Our induction hypothesis implies that G' has a desired vertex-minor.



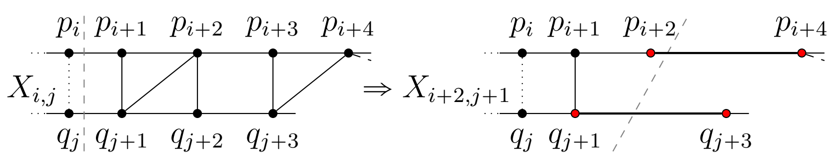
(a) Apply $G * p_{i+2} \backslash p_{i+2}$



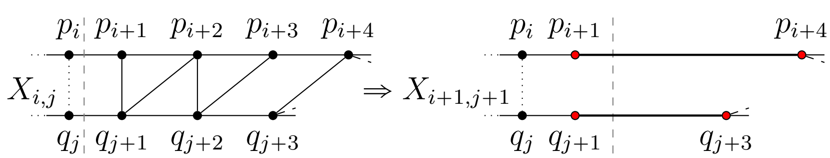
(b) Apply $G * p_{i+2} * q_{j+2} \backslash p_{i+2} \backslash q_{j+2}$



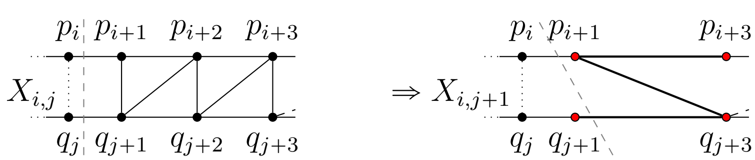
(c) Apply $G * q_{j+2} \backslash q_{j+2}$



(d) Apply $G * q_{j+2} * p_{i+3} \backslash q_{j+2} \backslash p_{i+3}$



(e) Apply $G \wedge p_{i+2}q_{j+2} * p_{i+3} \backslash p_{i+2} \backslash q_{j+2} \backslash p_{i+3}$



(f) Apply $G \wedge p_{i+2}q_{j+2} \backslash p_{i+2} \backslash q_{j+2}$

FIGURE 5. Cases in the proof of Lemma 4.6.

(f) If R has length at least 5, then $G' = G \wedge p_{i+2}q_{j+2} \backslash p_{i+2} \backslash q_{j+2}$ is a generalized ladder of maximum degree at most 4. Then $\delta_{G'}(X_{i,j+1})$ has exactly two edges and $\alpha_{i,j+1}(G') \geqslant \alpha_{i,j}(G) - 4$ and therefore $|X_{i,j+1}| + \alpha_{i,j+1}(G')/4 \geqslant |X_{i,j}| + 1 + (\alpha_{i,j}(G) - 4)/4 = |X_{i,j}| + \alpha_{i,j}(G)/4$. Our induction hypothesis implies that G' has a desired vertex-minor.

In all cases, we find the desired vertex-minor H. This completes the proof of Claim 1. □

Lemma 4.7. *Let G be a generalized ladder of maximum degree at most 4. If $|V(G)| \geqslant 192n^3$, then G has a cycle of length $4n+3$ as a vertex-minor.*

Proof. Let P, Q be two defining paths of G. We may assume $a > 1$ and $b > 1$ because $(192n^3 - 2)/3 + 2 \geqslant 4n + 3$.

Let p_xq_y be the unique chord other than p_1q_1 with minimum $x + y$. We claim that we may assume $(x-1)+(y-1) \leqslant 2$. Suppose not. Then p_xq_y, p_1q_1 and subpaths of P and Q form a cycle of length $x + y \geqslant 5$ and $p_1, p_2, \ldots, p_{x-1}, q_1, q_2, \ldots, q_{y-1}$ have degree 2. By moving the first few vertices of P to Q or Q to P, we may assume that $x \geqslant 3$ and $y \geqslant 2$. Then we may replace G with $G * p_1$. This proves the claim.

Thus the induced cycle containing p_1q_1 has at most 2 edges from $E(P) \cup E(Q)$. Similarly we may assume that the induced cycle containing p_aq_b has at most 2 edges from $E(P) \cup E(Q)$.

If G has at least $48n^2$ vertices of degree 3 or 4, then by Lemma 4.6, G has a generalized ladder H as a vertex-minor such that $|V(H)| \geqslant 12n^2$ and H has maximum degree at most 3. By Lemma 4.5, H has a cycle of length $4n + 3$ as a vertex-minor.

Thus we may assume that G has less than $48n^2$ vertices of degree 3 or 4. We may assume that G has at least one vertex of degree at least 3. The cycle formed by edges in $E(P) \cup E(Q) \cup \{p_1q_1, p_aq_b\}$ is partitioned into less than $48n^2$ paths whose internal vertices have degree 2 in G. One of the paths has length greater than $192n^3/(48n^2) = 4n$. Then there is an induced cycle C of G containing this path. Since C does not contain p_1q_1 or p_aq_b, C must contain two edges not in $E(P) \cup E(Q) \cup \{p_1q_1, p_aq_b\}$. Thus the length of C is at least $4n+3$. □

4.4. **Treating all generalized ladders.**

Lemma 4.8. *Let G be a generalized ladder. If G has n vertices of degree at least 4, then G has a vertex-minor H that is a generalized ladder such that the maximum degree of H is at most 4 and H has at least n vertices.*

Proof. Let S be the set of vertices having degree at least 4. For each vertex v in S, let P_v be the minimal subpath of Q containing all neighbors of v in Q if $v \in V(P)$ and let P_v be the minimal subpath of P containing all neighbors of v in P if $v \in V(Q)$.

Then each internal vertex of P_v has degree 2 or 3 and has degree 3 if and only if it is adjacent to v. We apply local complementation to each internal vertex and delete all internal vertices of P_v. It is easy to see that the resulting graph H is a generalized ladder and moreover $S \subseteq V(H)$ and every vertex in S has degree at most 4 in H. □

We are now ready to prove the main proposition of this section.

Proof of Proposition 4.1. Let G be such a graph. If G has at least $192n^3$ vertices of degree at least 4, then by Lemma 4.8, G has a vertex-minor H having at least $192n^3$ vertices such that H is a generalized ladder of maximum degree at most 4. By Lemma 4.7, H has a cycle of length $4n+3$ as a vertex-minor.

Thus we may assume that G has less than $192n^3$ vertices of degree at least 4. For a vertex v in P having degree at least 5, let q_i, q_j be two neighbors of v in Q such that if q_k is a neighbor of v in Q, then $i \leqslant k \leqslant j$. By Lemma 4.3, if $j-i+2 \geqslant 24n^2-3$, then G contains a cycle of length $4n+3$ as a vertex-minor. Thus we may assume $j-i \leqslant 24n^2-6$. The subpath of Q from q_i to q_j contains $j-i-1 \leqslant 24n^2-7$ internal vertices. Similarly the same bound holds for a vertex v in Q having degree at least 5. As in the proof of Lemma 4.8, we apply local complementation and delete all internal vertices of the minimal path spanning the neighbors of each vertex of degree at least 5 to obtain H. Then each vertex of degree at least 5 in G will have degree at most 4 in H. Since we remove at most $(192n^3-1)(24n^2-7)$ vertices,

$$|V(H)| \geqslant |V(G)| - (192n^3-1)(24n^2-7) > 192n^3.$$

By Lemma 4.7, H has a cycle of length $4n+3$ as a vertex-minor. □

5. Blocking sequences

Let A, B be two disjoint subsets of the vertex set of a graph G. By the definition of ρ^*_G and ρ_G, it is clear that

$$\text{if } A \subseteq X \subseteq V(G)\setminus B, \text{ then } \rho^*_G(A,B) \leqslant \rho_G(X).$$

What prevents us to achieve the equality for some X? We now present a tool called a blocking sequence, that is a certificate to guarantee that no such X exists. Blocking sequences were introduced by Geelen [13].

A sequence $v_1, v_2, \ldots, v_m$ ($m \geqslant 1$) is called a *blocking sequence* of a pair (A, B) of disjoint subsets A, B of $V(G)$ if

(a) $\rho^*_G(A, B \cup \{v_1\}) > \rho^*_G(A, B)$,
(b) $\rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}\}) > \rho^*_G(A, B)$ for all $i = 1, 2, \ldots, m-1$,
(c) $\rho^*_G(A \cup \{v_m\}, B) > \rho^*_G(A, B)$,
(d) no proper subsequence of $v_1, \ldots, v_m$ satisfies (a), (b), and (c).

The condition (d) is essential for the following standard lemma.

Lemma 5.1. *Let $v_1, v_2, \ldots, v_m$ be a blocking sequence for (A, B) in a graph G. Let X, Y be disjoint subsets of $\{v_1, v_2, \ldots, v_m\}$ such that if $v_i \in X$ and $v_j \in Y$, then $i < j$. Then*

$$\rho^*_G(A \cup X, B \cup Y) = \rho^*_G(A, B)$$

if and only if $v_1 \notin Y$, $v_m \notin X$, and for all $i \in \{1, 2, \ldots, m-1\}$, either $v_i \notin X$ or $v_{i+1} \notin Y$.

Proof. The forward direction is trivial. Let us prove the backward implication. Let $k = \rho^*_G(A, B)$. It is enough to prove $\rho^*_G(A \cup X, B \cup Y) \leqslant k$. Suppose that $v_1 \notin Y$, $v_m \notin X$, and for all $i \in \{1, 2, \ldots, m-1\}$, either $v_i \notin X$ or $v_{i+1} \notin Y$ and yet $\rho^*_G(A \cup X, B \cup Y) > k$. We may assume that $|X| + |Y|$ is chosen to be minimum. If $|X| \geqslant 2$, then we can partition X into two nonempty sets X_1 and X_2. Then by the hypothesis, $\rho^*_G(A \cup X_1, B \cup Y) = \rho^*_G(A \cup X_2, B \cup Y) = k$. By Lemma 2.2, we deduce that $\rho^*_G(A \cup X_1, B \cup Y) + \rho^*_G(A \cup X_2, B \cup Y) \geqslant k + \rho^*_G(A \cup X, B \cup Y)$ and therefore we deduce that $\rho^*_G(A \cup X, B \cup Y) \leqslant k$. So we may assume $|X| \leqslant 1$. By symmetry we may also assume $|Y| \leqslant 1$. Then by the condition (d), this is clear. □

The following proposition states that a blocking sequence is a certificate that $\rho_G(X) > \rho^*_G(A, B)$ for all $A \subseteq X \subseteq V(G) \backslash B$. This appears in almost all applications of blocking sequences. The proof uses the submodular inequality (Lemma 2.2).

Proposition 5.2 (Geelen [13, Lemma 5.1]; see Oum [17])**.** *Let G be a graph and A, B be two disjoint subsets of $V(G)$. Then G has a blocking sequence for (A, B) if and only if $\rho_G(X) > \rho^*_G(A, B)$ for all $A \subseteq X \subseteq V(G) \backslash B$.*

The following proposition allows us to change the graph to reduce the length of a blocking sequence. This was pointed out by Geelen [private communication with the second author, 2005]. A special case of the following proposition is presented in [17].

Proposition 5.3. *Let G be a graph and A, B be disjoint subsets of $V(G)$. Let $v_1, v_2, \ldots, v_m$ be a blocking sequence for (A,B) in G. Let $1 \leqslant i \leqslant m$.*

- *If $m > 1$, then $\rho^*_{G*v_i}(A,B) = \rho^*_G(A,B)$ and a sequence*

$$v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$$

obtained by removing v_i from the blocking sequence is a blocking sequence for (A,B) in $G * v_i$.

- *If $m = 1$, then $\rho^*_{G*v_i}(A,B) = \rho^*_G(A,B) + 1$.*

Proof. Let $k = \rho^*_G(A,B)$ and $H = G * v_i$.

If $m = 1$, then by Lemma 2.5,

$$\rho^*_H(A,B) + \rho^*_G(A,B) \geqslant \rho^*_G(A \cup \{v_1\}, B) + \rho^*_G(A, B \cup \{v_1\}) - 1 \geqslant 2k + 1$$

and therefore $\rho^*_H(A,B) \geqslant k+1$. Since $\rho^*_H(A,B) \leqslant \rho^*_H(A, B \cup \{v_1\}) = \rho^*_G(A, B \cup \{v_1\}) \leqslant k+1$, we deduce that $\rho^*_H(A,B) = k+1$ if $m = 1$.

Now we assume that $m \neq 1$. First it is easy to observe that $\rho^*_H(X,Y) \leqslant \rho^*_G(X, Y \cup \{v_i\})$ and $\rho^*_H(X,Y) \leqslant \rho^*_G(X \cup \{v_i\}, Y)$ whenever X, Y are disjoint subsets of $V(G) \setminus \{v_i\}$, because the local complementation does not change the cut-rank function of $G[X \cup Y \cup \{v_i\}]$. This with Lemma 5.1 implies that

- $\rho^*_H(A,B) \leqslant k$,
- $\rho^*_H(A \cup \{v_j\}, B) \leqslant k$ for all $j \in \{1,2,\ldots,m\} \setminus \{i-1, m\}$,
- $\rho^*_H(A \cup \{v_{i-1}\}, B) \leqslant k$ if $i \neq 1, m$.
- $\rho^*_H(A, B \cup \{v_j\}) \leqslant k$ for all $j \in \{1,2,\ldots,m\} \setminus \{1, i+1\}$.
- $\rho^*_H(A, B \cup \{v_{i+1}\}) \leqslant k$ if $i \neq 1, m$.
- $\rho^*_H(A \cup \{v_j\}, B \cup \{v_\ell\}) \leqslant k$ for all $j, \ell \in \{1,2,\ldots,m\} \setminus \{i\}$ with $\ell - j > 1$, unless $j + 1 = i = \ell - 1$.

Let $B' = B \cup \{v_{i+1}\}$ if $i < m$ and $B' = B$ otherwise. Then $\rho^*_G(A \cup \{v_i\}, B') = k+1$ and $\rho^*_G(A, B') = k$.

(1) We claim that if $i > 1$, then $\rho^*_H(A, B \cup \{v_1\}) > k$. By Lemma 2.5,

$$\rho^*_H(A, B' \cup \{v_1\}) + \rho^*_G(A, B') \geqslant \rho^*_G(A, B' \cup \{v_1, v_i\}) + \rho^*_G(A \cup \{v_i\}, B') - 1,$$

and therefore we deduce that $\rho^*_H(A, B' \cup \{v_1\}) \geqslant \rho^*_G(A, B' \cup \{v_1, v_i\}) > k$. By Lemma 2.2, $\rho^*_H(A, B' \cup \{v_i\}) + \rho^*_H(A, B \cup \{v_1\}) \geqslant \rho^*_H(A, B' \cup \{v_1, v_i\}) + \rho^*_H(A,B) > 2k$. We deduce that $\rho^*_H(A, B \cup \{v_1\}) > k$ because $\rho^*_H(A, B' \cup \{v_i\}) = \rho^*_G(A, B' \cup \{v_i\}) = k$ by Lemma 5.1.

(2) By (1) and symmetry between A and B, if $i < m$, then $\rho^*_H(A \cup \{v_m\}, B) > k$.

Then we deduce that $\rho^*_H(A,B) \geqslant k$ and therefore $\rho^*_H(A,B) = k$.

(3) We claim that if $j < i-1$, then $\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) > k$. By Lemma 2.5,

$$\begin{aligned}&\rho^*_H(A \cup \{v_j\}, B' \cup \{v_{j+1}\}) + \rho^*_G(A \cup \{v_j\}, B')\\ &\qquad\geqslant \rho^*_G(A \cup \{v_j\}, B' \cup \{v_{j+1}, v_i\}) + \rho^*_G(A \cup \{v_j, v_i\}, B') - 1 > 2k,\end{aligned}$$

and therefore $\rho^*_H(A \cup \{v_j\}, B' \cup \{v_{j+1}\}) > k$. By Lemma 2.2, $\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) + \rho^*_H(A \cup \{v_j\}, B') \geqslant \rho^*_H(A \cup \{v_j\}, B' \cup \{v_{j+1}\}) + \rho^*_H(A \cup \{v_j\}, B) > 2k$. Note that $\rho^*_H(A \cup \{v_j\}, B) \geqslant \rho^*_H(A, B) = k$. Since $\rho^*_H(A \cup \{v_j\}, B') \leqslant \rho^*_H(A \cup \{v_j\}, B' \cup \{v_i\}) = \rho^*_G(A \cup \{v_j\}, B' \cup \{v_i\}) \leqslant k$, we deduce that $\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) > k$.

(4) By symmetry, we deduce from (3) that if $i < j < m$, then $\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) > k$.

(5) We claim that $\rho^*_H(A \cup \{v_{i-1}\}, B') > k$. By Lemma 2.5,

$$\begin{aligned}&\rho^*_H(A \cup \{v_{i-1}\}, B') + \rho^*_G(A \cup \{v_{i-1}\}, B')\\ &\qquad\geqslant \rho^*_G(A \cup \{v_{i-1}\}, B' \cup \{v_i\}) + \rho^*_G(A \cup \{v_{i-1}, v_i\}, B') - 1 > 2k.\end{aligned}$$

Since $\rho^*_G(A \cup \{v_{i-1}\}, B') = k$, we have $\rho^*_H(A \cup \{v_{i-1}\}, B') > k$.

This completes the proof of the lemma that $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$ is a blocking sequence of (A, B) in $G * v_i$. □

Corollary 5.4. *Let G be a graph and A, B be disjoint subsets of $V(G)$. Let $v_1, v_2, \ldots, v_m$ be a blocking sequence for (A, B) in G. Let $1 \leqslant i \leqslant m$. Suppose that v_i has a neighbor w in $A \cup B$.*

- *If $m > 1$, then $\rho^*_{G \wedge v_i w}(A, B) = \rho^*_G(A, B)$ and the sequence $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$ obtained by removing v_i from the blocking sequence is a blocking sequence for (A, B) in $G \wedge v_i w$.*
- *If $m = 1$, then $\rho^*_{G \wedge v_i w}(A, B) = \rho^*_G(A, B) + 1$.*

Proof. It follows easily from the facts that $G \wedge v_i w = G * w * v_i * w$ and $\rho^*_G(X, Y) = \rho^*_{G*w}(X, Y)$ for all graphs G with $w \in X \cup Y$. □

Corollary 5.5. *Let G be a graph and A, B be disjoint subsets of $V(G)$. Let $v_1, v_2, \ldots, v_m$ be a blocking sequence for (A, B) in G. Let $1 \leqslant i \leqslant m$. Suppose that v_i and $v_{i'}$ are adjacent and $i < i'$.*

- *If $m > 2$, then $\rho^*_{G \wedge v_i v_{i'}}(A, B) = \rho^*_G(A, B)$ and the sequence $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i'-1}, v_{i'+1}, \ldots, v_m$ obtained by removing v_i and $v_{i'}$ from the blocking sequence is a blocking sequence for (A, B) in $G \wedge v_i v_{i'}$.*
- *If $m = 2$, then $\rho^*_{G \wedge v_i v_{i'}}(A, B) = \rho^*_G(A, B) + 1$.*

Proof. If v_i has a neighbor w in $A \cup B$, then $G \wedge v_i v_{i'} = G \wedge v_i w \wedge w v_{i'}$ and this corollary follows from Corollary 5.4. So we may assume that v_i

has no neighbors in $A \cup B$ and similarly $v_{i'}$ has no neighbors in $A \cup B$. Thus $i, i' \notin \{1, m\}$ and $m \geqslant 4$.

Since v_i and $v_{i'}$ are adjacent, we may assume that $i' = i+1$. Let $H = G \wedge v_i v_{i+1}$ and $k = \rho^*_G(A, B)$. Since v_i and v_{i+1} have no neighbors in $A \cup B$, $\rho^*_H(A, B) = k$.

Then $v_1, v_2, \ldots, v_i$ is a blocking sequence for $(A, B \cup \{v_{i+1}\})$ in G by Lemma 5.1. Similarly $v_{i+1}, v_{i+2}, \ldots, v_m$ is a blocking sequence for $(A \cup \{v_i\}, B)$ in G.

By Corollary 5.4, $v_1, v_2, \ldots, v_{i-1}$ is a blocking sequence for $(A, B \cup \{v_{i+1}\})$ in H. Then $\rho^*_H(A, B \cup \{v_1\}) = \rho^*_H(A, B \cup \{v_1, v_{i+1}\}) > k$, because v_{i+1} has no neighbors of H in A.

For $1 \leqslant j < i-1$, $\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) + \rho^*_H(A \cup \{v_j\}, B \cup \{v_{i+1}\}) \geqslant \rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}, v_{i+1}\}) + \rho^*_H(A \cup \{v_j\}, B) > 2k$ and therefore

$$\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) > k$$

because $\rho^*_H(A \cup \{v_j\}, B) \leqslant \rho^*_H(A \cup \{v_j\}, B \cup \{v_{i+1}\}) \leqslant k$.

Similarly $v_{i+2}, v_{i+3}, \ldots, v_m$ is a blocking sequence for $(A \cup \{v_i\}, B)$ in H. By symmetry, we deduce that $\rho^*_H(A \cup \{v_m\}, B) > k$ and $\rho^*_H(A \cup \{v_j\}, B \cup \{v_{j+1}\}) > k$ for all $i+1 < j < m$.

We now claim that $\rho^*_H(A \cup \{v_{i-1}\}, B \cup \{v_{i+2}\}) > k$. By Lemma 2.2,

$$\begin{aligned}\rho^*_H(A \cup \{v_{i-1}\}, B \cup \{v_{i+2}\}) &+ \rho^*_H(A \cup \{v_{i+1}\}, B \cup \{v_{i+2}\}) \\ &\geqslant \rho^*_H(A \cup \{v_{i-1}, v_{i+1}\}, B \cup \{v_{i+2}\}) + \rho^*_H(A, B \cup \{v_{i+2}\}).\end{aligned}$$

Since v_{i+1} has no neighbors in $A \cup B$, we have $\rho^*_H(A \cup \{v_{i+1}\}, B \cup \{v_{i+2}\}) = \rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+2}\}) = k$ and $\rho^*_H(A, B \cup \{v_{i+2}\}) = \rho^*_G(A, B \cup \{v_{i+2}\}) = k$. Therefore

$$\rho^*_H(A \cup \{v_{i-1}\}, B \cup \{v_{i+2}\}) \geqslant \rho^*_H(A \cup \{v_{i-1}, v_{i+1}\}, B \cup \{v_{i+2}\}).$$

By Lemma 2.5,

$$\begin{aligned}\rho^*_H(A \cup \{v_{i-1}, v_{i+1}\}, B \cup \{v_{i+2}\}) &+ \rho^*_G(A \cup \{v_{i-1}\}, B \cup \{v_{i+1}, v_{i+2}\}) \\ &\geqslant \rho^*_G(A \cup \{v_{i-1}\}, B \cup \{v_i, v_{i+1}, v_{i+2}\}) \\ &\quad + \rho^*_G(A \cup \{v_{i-1}, v_i, v_{i+1}\}, B \cup \{v_{i+2}\}) - 1.\end{aligned}$$

By Lemma 5.1, $\rho^*_G(A \cup \{v_{i-1}, v_i, v_{i+1}\}, B \cup \{v_{i+2}\}) > k$ and $\rho^*_G(A \cup \{v_{i-1}\}, B \cup \{v_{i+1}, v_{i+2}\}) = k$. Therefore $\rho^*_H(A \cup \{v_{i-1}\}, B \cup \{v_{i+2}\}) \geqslant \rho^*_H(A \cup \{v_{i-1}, v_{i+1}\}, B \cup \{v_{i+2}\}) \geqslant \rho^*_G(A \cup \{v_{i-1}\}, B \cup \{v_i, v_{i+1}, v_{i+2}\}) > k$. This proves the claim.

So far we have shown that the sequence $v_1, v_2, \ldots, v_{i-1}, v_{i+2}, \ldots, v_m$ satisfies (a), (b), (c) of the definition of blocking sequences. It remains to show (d). For $j \in \{2, 3, \ldots, m\} \setminus \{i, i+1\}$, $\rho^*_H(A, B \cup \{v_j\}) = \rho^*_G(A, B \cup \{v_j\}) = k$ because v_i and v_{i+1} have no neighbors in $A \cup B$. Similarly

$\rho^*_H(A\cup\{v_j\},B) = \rho^*_G(A\cup\{v_j\},B) = k$ for $j \in \{1,2,\dots,m-1\}\setminus\{i,i+1\}$. For $j,\ell \in \{1,2,\dots,m\}\setminus\{i,i+1\}$ with $\ell - j > 1$, either $\rho^*_G(A\cup\{v_j\}, B\cup\{v_\ell, v_i, v_{i+1}\}) = k$ or $\rho^*_G(A\cup\{v_j,v_i,v_{i+1}\}, B\cup\{v_\ell\}) = k$ and therefore $\rho^*_H(A\cup\{v_j\}, B\cup\{v_\ell\}) \leqslant k$, unless $j = i-1$ and $\ell = i+2$. This completes the proof. □

We will now prove that without loss of generality, a blocking sequence for (A,B) is short by applying local complementation while keeping the subgraph induced on $A\cup B$.

Proposition 5.6. *Let G be a prime graph and let A, B be disjoint subsets of $V(G)$ with $|A|,|B| \geqslant 2$. Suppose that there exist two nonempty sets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that the set of all edges between A and B is $\{xy : x\in A_0, y\in B_0\}$. Let*

$$\ell_0 = \begin{cases} 3 & \text{if } |A_0| = |B_0| = 1, \\ 4 & \text{if } |A_0| = 1 \text{ or } |B_0| = 1, \\ 6 & \text{otherwise.} \end{cases}$$

Then there exists a graph G' locally equivalent to G satisfying the following.

(i) $G[A\cup B] = G'[A\cup B]$.
(ii) G' has a blocking sequence $b_1, b_2, \dots, b_\ell$ of length at most ℓ_0 for (A,B).

Proof. Since G is prime, G has a blocking sequence for (A,B) by Proposition 5.2. Let $\mathcal{G}$ be the set of all graphs G' locally equivalent to G such that $G'[A\cup B] = G[A\cup B]$. We assume that G is chosen in $\mathcal{G}$ so that the length ℓ of a blocking sequence $b_1, b_2, \dots, b_\ell$ for (A,B) is minimized.

For $1 \leqslant i < \ell$, $N_G(b_i)\cap B = B_0$ or $\varnothing$ because $\rho_G(A\cup\{b_i\}, B) = 1$. For $1 < i \leqslant \ell$, $N_G(b_i)\cap A = A_0$ or $\varnothing$ because $\rho_G(A, B\cup\{b_i\}) = 1$.

Suppose that $N_G(b_i)\cap(A\cup B) = N_G(b_j)\cap(A\cup B)$ for some $1 < i < j < \ell$. If b_i and b_j are adjacent, then $G' = G\wedge b_ib_j \in \mathcal{G}$. If b_i and b_j are non-adjacent, then $G' = G * b_i * b_j \in \mathcal{G}$. In both cases, we found a graph in $\mathcal{G}$ having a shorter blocking sequence by Proposition 5.3 or Corollary 5.5, contradicting our assumption.

If $|B_0| = 1$, then for all $1 < i < \ell$, $N_G(b_i)\cap A = A_0$ because otherwise $G * b_i \in \mathcal{G}$ has a shorter blocking sequence by Proposition 5.3, contradicting our assumption. Similarly if $|A_0| = 1$, then $N_G(b_i)\cap B = B_0$ for all $1 < i < \ell$.

By the pigeonhole principle, we deduce that $\ell \leqslant \ell_0$. □

6. Obtaining a long cycle from a huge induced path

In this section we aim to prove the following theorem.

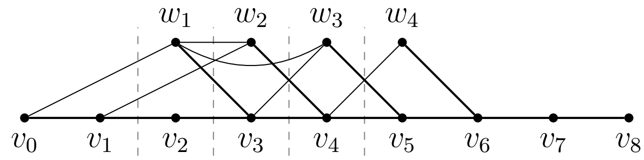


FIGURE 6. An example of a 4-patched path of length 8.

Theorem 6.1. *If a prime graph has an induced path of length* $\lceil 6.75n^7 \rceil$, *then it has a cycle of length* n *as a vertex-minor.*

The main idea is to find a big generalized ladder, defined in Section 4 as a vertex-minor by using blocking sequences in Section 5.

6.1. **Patching a path.** For $1 \leqslant k \leqslant n-2$, a *$k$-patch* of an induced path $P = v_0 v_1 \cdots v_n$ of a graph G is a sequence $Q = w_1, w_2, \ldots, w_k$ of distinct vertices not on P such that for each $i \in \{1, 2, \ldots, k\}$,

(i) v_{i+2} is the only vertex adjacent to w_i among $v_{i+1}, v_{i+2}, \ldots, v_n$,
(ii) $\varnothing \neq N_G(w_i) \cap \{v_0, \ldots, v_i, w_1, \ldots, w_{i-1}\} \neq \{v_i, w_{i-1}\}$ if $i > 1$,
(iii) $N_G(w_1) \cap \{v_0, v_1\} = \{v_0\}$.

An induced path is called *k-patched* if it has a k-patch. An induced path of length n is called *fully patched* if it is equipped with a $(n-2)$-patch. See Figure 6 for an example.

Our goal is to find a fully patched long induced path in a vertex-minor of a prime graph having a very long induced path.

Lemma 6.2. *Let $P = v_0 v_1 \ldots v_m$ be an induced path from $s = v_0$ to $t = v_m$ in a graph G and let H be a connected induced subgraph of $G \backslash V(P)$. Let v be a vertex in $V(G) \backslash (V(H) \cup V(P))$. Suppose that $N_G(V(H)) \cap V(P) = \{s\}$, $|E(P)| \geqslant 6(n-1)^2 - 5$, and v has neighbors in both $V(P) \backslash \{s\}$ and $V(H)$.*

If G has no cycle of length $2n+1$ as a vertex-minor, then there exist a graph G' locally equivalent to G and an induced path P' from s to t of G' disjoint from $V(H)$ satisfying the following.

(i) $G[V(H) \cup \{s\}] = G'[V(H) \cup \{s\}]$,
(ii) $N_G(v) \cap V(H) = N_{G'}(v) \cap V(H)$,
(iii) $P' = v_0 v_i v_{i+1} v_{i+2} \cdots v_m$ *for some* i,
(iv) v_i *is the only vertex on* $V(P')$ *adjacent to* v *in* G',
(v) $|E(P')| \geqslant |E(P)| - 6(n-1)^2 + 6$.

Proof. Since G has a cycle using H with s and P, G is not a forest and therefore $n \geqslant 2$. Let $v_0 = s, v_1, v_2, \ldots, v_m = t$ be vertices in P. Let v_k be the neighbor of v with maximum k. Then G has a fan having at least $k+3$ vertices because H is connected and v has a neighbor in H.

If $k \geqslant 6(n-1)^2 - 6$, then G has a fan having at least $6(n-1)^2 - 3$ vertices and by Lemma 4.3, G contains a cycle of length $2n+1$ as a vertex-minor. This contradicts to our assumption that G has no such vertex-minor. Thus, $k \leqslant 6(n-1)^2 - 7$.

Let $G_0 = G * v_1 * v_2 * v_3 \cdots * v_{k-2}$ and let $P_0 = v_0 v_{k-1} v_k v_{k+1} \cdots v_m$. (If $k \leqslant 2$, then let $G_0 = G$ and $P_0 = P$.) Then clearly P_0 is an induced path of G_0 and $v_k \in N_{G_0}(v) \cap V(P_0) \subseteq \{v_0, v_{k-1}, v_k\}$.

If $N_{G_0}(v) \cap V(P_0) = \{v_k\}$, then we are done by taking $G' = G_0 * v_{k-1}$ and $P' = v_0 v_k v_{k+1} \cdots v_m$.

If $N_{G_0}(v) \cap V(P_0) = \{v_{k-1}, v_k\}$, then we can take $G' = G_0 * v_k * v_{k-1}$ and $P' = v_0 v_{k+1} v_{k+2} \cdots v_m$.

If $N_{G_0}(v) \cap V(P_0) = \{v_0, v_k\}$, then we can take $G' = G_0 * v_{k-1} * v_k$ and $P' = v_0 v_{k+1} v_{k+2} \cdots v_m$.

Finally, if $N_{G_0}(v) \cap V(P_0) = \{v_0, v_{k-1}, v_k\}$, then we can take $G' = G_0 * v_k * v_{k-1} * v_{k+1}$ and $P' = v_0 v_{k+2} v_{k+3} \cdots v_m$.

In all cases, $|E(P')| \geqslant |E(P)| - (k+1) \geqslant |E(P)| - 6(n-1)^2 + 6$. ☐

Lemma 6.3. *Let $n \geqslant 2$. Let G be a prime graph having an induced path of length t. If $t \geqslant 6(n-1)^2 - 3$, then there exists a graph G' locally equivalent to G having a 1-patched induced path of length $t - 6(n-1)^2 + 6$, unless G has a cycle of length $2n+1$ as a vertex-minor.*

Proof. We may choose G so that the length t of an induced path P is maximized among all graphs locally equivalent to G. Let $v_0, v_1, \ldots, v_m$ be vertices of P in this order. Since G is prime, v_0 has a neighbor v other than v_1. We may assume that v is non-adjacent to v_1 because otherwise we can replace G with $G * v_0$.

Since P is a longest induced path, v must have some neighbors in $V(P) \backslash \{v_0, v_1\}$. We now apply Lemma 6.2 with $H = G[\{v_0, v_1\}]$, deducing that there exists a graph G' locally equivalent to G having a 1-patched induced path of length $t - 6(n-1)^2 + 6$, unless G has a cycle of length $2n+1$ as a vertex-minor. ☐

Lemma 6.4. *Let $n \geqslant 2$. Let G be a prime graph and let P be a k-patched induced path $v_0 v_1 \cdots v_t$. If $t \geqslant 6(n-1)^2 + k$, then there exists a graph G' locally equivalent to G having a $(k+1)$-patched induced path $v_0 v_1 \cdots v_{k+2} v_i v_{i+1} \cdots v_t$ of length at least $t - 6(n-1)^2 + 3$ with some $i > k+2$, unless G has a cycle of length $2n+1$ as a vertex-minor.*

Proof. Let $P = v_0 v_1 \ldots v_t$ be an induced path of length t in G and $Q = w_1, w_2, \ldots, w_k$ be its k-patch. Suppose that G has no vertex-minor isomorphic to a cycle of length $2n+1$.

Let $A = \{v_0, v_1, \ldots, v_{k+1}\} \cup Q$. By Proposition 5.6, we may assume that G has a blocking sequence $b_1, b_2, \ldots, b_\ell$ of length at most 4 for

$(A, V(P)\backslash A)$ because v_{k+2} is the only vertex in $V(P)\backslash A$ having neighbors in A.

Notice that $P\backslash A$ is an induced path of G. We say that a blocking sequence $b_1, b_2, \ldots, b_\ell$ for $(A, V(P)\backslash A)$ is *nice* if b_ℓ has a unique neighbor in $V(P)\backslash A$, that is also a unique neighbor of v_{k+2} in $V(P)\backslash A$.

We know that b_ℓ has neighbors in $\{v_{k+3}, \ldots, v_t\}$ by the definition of a blocking sequence. We take $H = G[A \cup Q \cup \{b_1, b_2, \ldots, b_{\ell-1}\}]$. By Lemma 6.2, there exist a graph G_ℓ locally equivalent to G and an induced path $P_\ell = v_0v_1 \cdots v_{k+2}v_iv_{i+1} \cdots v_t$ of G_ℓ for some i with a k-patch Q such that $G_\ell[A \cup \{v_{k+2}\}] = G[A \cup \{v_{k+2}\}]$, a sequence $b_1, b_2, \ldots, b_\ell$ is a nice blocking sequence for $(A, V(P_\ell)\backslash A)$ in G_ℓ, and $|E(P_\ell)| \geqslant t - 6(n-1)^2 + 6$.

Let $r \geqslant 1$ be minimum such that there exist a graph G' locally equivalent to G and an induced path $P' = v_0v_1 \cdots v_{k+2}v_iv_{i+1} \cdots v_m$ for some i with a k-patch Q in G' such that $G'[A \cup \{v_{k+2}\}] = G[A \cup \{v_{k+2}\}]$, a sequence $b_1, b_2, \ldots, b_r$ is a nice blocking sequence for $(A, V(P')\backslash A)$ in G', and $|E(P')| \geqslant t - 6(n-1)^2 + 6 + r - \ell$. Such r exists because G_ℓ and P_ℓ satisfy the condition when $r = \ell$.

We claim that $r = 1$. Suppose $r > 1$.

Suppose that b_r is non-adjacent to v_{k+1} in G'. Then v_i is the only neighbor of b_r in $V(P')$ in G' and b_r is adjacent to b_{r-1} in G'. If b_{r-1} is non-adjacent to v_{k+2}, then take $G'' = G' * b_r$ and $P'' = P'$; in G'', a sequence $b_1, b_2, \ldots, b_{r-1}$ is a nice blocking sequence for $(A, V(P')\backslash A)$ and the length of P' is at least $t - 6(n-1)^2 + 6 + r - \ell$. This leads a contradiction to the assumption that r is minimized. Therefore b_{r-1} is adjacent to v_{k+2}. Then take $G'' = G' * b_r * v_i$ with $P'' = v_0v_1 \cdots v_{k+2}v_{i+1} \cdots v_m$. Then $b_1, b_2, \ldots, b_{r-1}$ is a nice blocking sequence for $(A, V(P'')\backslash A)$ in G'' and the length of P'' is at least $t - 6(n-1)^2 + 6 + r - \ell - 1$. This contradicts to the assumption that r is chosen to be minimum.

Therefore b_r is adjacent to v_{k+1} in G'. Since b_r is the last vertex in the blocking sequence, b_r is also adjacent to w_k in G'. If b_{r-1} is non-adjacent to v_{k+2}, then take $G'' = G' * v_{k+2} * b_r$ and $P'' = P'$; in G'', a sequence $b_1, b_2, \ldots, b_{r-1}$ is a nice blocking sequence for $(A, V(P'')\backslash A)$ and the length of P'' is at least $t - 6(n-1)^2 + 6 + r - \ell$, contradicting our assumption on r. So b_{r-1} is adjacent to v_{k+2}. Then we take $G'' = G' * v_{k+2} * b_r * v_i$ with $P'' = v_0v_1 \cdots v_{k+2}v_{i+1} \cdots v_m$. Then $b_1, b_2, \ldots, b_{r-1}$ is a nice blocking sequence for $(A, V(P'')\backslash A)$ in G'' and the length of P'' is at least $t - 6(n-1)^2 + 6 + r - \ell - 1$. This again contradicts to the assumption on r. This proves that $r = 1$.

Since b_1 is a nice blocking sequence for $(A, V(P')\backslash A)$ in G', b_1 has a neighbor in A in G' and $N_{G'}(b_1) \cap A \neq \{v_{k+1}, w_k\}$. In addition, v_i is the only neighbor of b_1 among $V(P')\backslash A$ in G'. Now it is easy to see that

$w_1, w_2, w_3, \ldots, w_k, b_1$ is a $(k+1)$-patch of P' in G'. And, since $\ell \leqslant 4$, we have $|E(P')| \geqslant t - 6(n-1)^2 + 3$. □

Proposition 6.5. *Let $N \geqslant 4$ be an integer. If a prime graph G on at least 5 vertices has an induced path of length $L = (6(n-1)^2 - 2)(N-2) - 1$, then there exists a graph G' locally equivalent to G having a fully patched induced path of length N, unless G has a cycle of length $2n+1$ as a vertex-minor.*

Proof. Suppose that G has no cycle of length $2n+1$ as a vertex-minor. Then $n \geqslant 3$ by Theorem 1.1. By Lemma 6.3, we may assume that G has a 1-patched path of length $L - 6(n-1)^2 + 6$. By Lemma 6.4, we may assume that G has an $(N-2)$-patched path of length

$$L - 6(n-1)^2 + 6 - (N-3)(6(n-1)^2 - 3) = N$$

Thus G has a fully patched induced path of length N. □

6.2. **Finding a cycle from a fully patched path.** We aim to find a cycle as a vertex-minor in a sufficiently long fully patched path.

Let $P = v_0 v_1 \cdots v_n$ be an induced path of a graph G with a $(n-2)$-patch $Q = w_1 w_2 w_3, \ldots w_{n-2}$. Let $A_1 = \{v_0, v_1\}$ and for $i = 2, \ldots, n-2$, let $A_i = \{v_0, v_1, \ldots, v_i, w_1, w_2, \ldots, w_{i-1}\}$ and $B_i = V(P) \backslash A_i$ for all $i \in \{1, 2, \ldots, n-2\}$.

For $i \geqslant 1$, let $L(w_i)$ be the minimum $j \geqslant 0$ such that

$$\rho^*_G(A_{j+1}, B_{j+1} \cup \{w_i\}) > 1.$$

Since w_i is a blocking sequence for (A_i, B_i), $L(w_i)$ is well defined and $L(w_i) < i$.

We classify vertices in Q as follows.

- A vertex w_i has *Type 0* if $L(w_i) = 0$ and w_i is adjacent to v_0.
- A vertex w_i has *Type 1* if $L(w_i) \geqslant 1$ and w_i has no neighbor in $A_{L(w_i)}$ and w_i is adjacent to exactly one of $v_{L(w_i)+1}$ and $w_{L(w_i)}$.
- A vertex w_i has *Type 2* if $L(w_i) = 1$ and w_i is adjacent to v_1, non-adjacent to v_0.
- A vertex w_i has *Type 3* if $L(w_i) \geqslant 2$ and w_i has no neighbor in $A_{L(w_i)-1}$ and w_i is adjacent to both $v_{L(w_i)}$ and $w_{L(w_i)-1}$.

By the definition of fully patched paths, we can deduce the following lemma easily.

Lemma 6.6. *Each vertex in Q has Type 0, 1, 2, or 3.*

Proof. If w_i is adjacent to v_0, then $\rho^*_G(A_1, B_1 \cup \{w_i\}) > 1$ and therefore $L(w_i) = 0$, implying that w_i has Type 0. We may now assume that w_i is non-adjacent to v_0 and so $L(w_i) > 0$.

If w_i has no neighbors in $A_{L(w_i)}$, then $\rho^*_G(A_{L(w_i)+1}, B_{L(w_i)+1} \cup \{w_i\}) = \rho^*_G(A_{L(w_i)+1} \backslash A_{L(w_i)}, B_{L(w_i)+1} \cup \{w_i\}) > 1$. Thus $v_{L(w_i)+2}$ and w_i cannot have the same set of neighbors in $A_{L(w_i)+1} \backslash A_{L(w_i)} = \{v_{L(w_i)+1}, w_{L(w_i)}\}$. By the definition of fully patched paths, $v_{L(w_i)+2}$ is adjacent to both $v_{L(w_i)+1}$ and $w_{L(w_i)}$. It follows that w_i is adjacent to exactly one of $v_{L(w_i)+1}$ and $w_{L(w_i)}$. So w_i has Type 1.

Now we may assume that w_i has some neighbors in $A_{L(w_i)}$. By definition,

$$\rho^*_G(A_{L(w_i)}, B_{L(w_i)} \cup \{w_i\}) \leqslant 1$$

and therefore w_i and $v_{L(w_i)+1}$ have the same set of neighbors in $A_{L(w_i)}$. Therefore, if $L(w_i) = 1$, then w_i is adjacent to v_1, implying that w_i has Type 2. If $L(w_i) > 1$, then w_i is adjacent to both $v_{L(w_i)}$ and $w_{L(w_i)-1}$, and so w_i has Type 3. □

We say that a pair of paths P^i_1 and P^i_2 from $\{v_0, v_1\}$ to $\{v_{i+1}, w_i\}$ is *good* if

(i) P^i_1 and P^i_2 are vertex-disjoint induced paths on A_{i+1},
(ii) for each $j \in \{1, 2, \ldots, i-1\}$, $w_j \in V(P^i_1) \cup V(P^i_2)$ or $v_{j+1} \in V(P^i_1) \cup V(P^i_2)$,
(iii) $G[V(P^i_1) \cup V(P^i_2)] + v_{i+1}w_i$ is a generalized ladder with two defining paths P^i_1 and P^i_2.

Lemma 6.7. *For all $i \in \{1, 2, \ldots, n-2\}$, G has a good pair of paths P^i_1 and P^i_2 from $\{v_0, v_1\}$ to $\{v_{i+1}, w_i\}$.*

Proof. We proceed by induction on i. If w_i has Type 0, then let $P^i_1 = v_1 v_2 \cdots v_{i+1}$ and $P^i_2 = v_0 w_i$. Since v_0 has no neighbors in $\{v_2, v_3, \ldots, v_{i+1}\}$, $G[V(P^i_1) \cup V(P^i_2)] + v_{i+1}w_i$ is a generalized ladder with two defining paths P^i_1 and P^i_2. Also, $V(P^i_1) \cup V(P^i_2) \subseteq A_{i+1}$ and for all $j \in \{1, 2, \ldots, i-1\}$, $v_{j+1} \in V(P^i_1)$. Thus, the pair (P^i_1, P^i_2) is good.

If w_i has Type 2, then let $P^i_1 = v_0 w_1 v_3 v_4 \cdots v_{i+1}$ and $P^i_2 = v_1 w_i$. By the definition of a patched path, v_1 is not adjacent to w_1. So, v_1 has no neighbors in $\{w_1, v_3, v_4, \ldots, v_{i+1}\}$, and therefore $G[V(P^i_1) \cup V(P^i_2)] + v_{i+1}w_i$ is a generalized ladder with two defining paths P^i_1 and P^i_2. Clearly, $V(P^i_1) \cup V(P^i_2) \subseteq A_{i+1}$. Moreover, $w_1 \in V(P^i_1)$ and for each $j \in \{2, \ldots, i-1\}$, $v_{j+1} \in V(P^i_1)$. Therefore, the pair (P^i_1, P^i_2) is good.

Now, we may assume that w_i has Type 1 or Type 3. Since $L(w_i) \geqslant 1$, by the induction hypothesis, G has a good pair of paths $P^{L(w_i)}_1$, $P^{L(w_i)}_2$ from $\{v_0, v_1\}$ to $\{v_{L(w_i)+1}, w_{L(w_i)}\}$.

Suppose w_i has Type 1 and therefore w_i is adjacent to exactly one of $v_{L(w_i)+1}$ and $w_{L(w_i)}$. Let $\{x, y\} = \{v_{L(w_i)+1}, w_{L(w_i)}\}$ such that x is

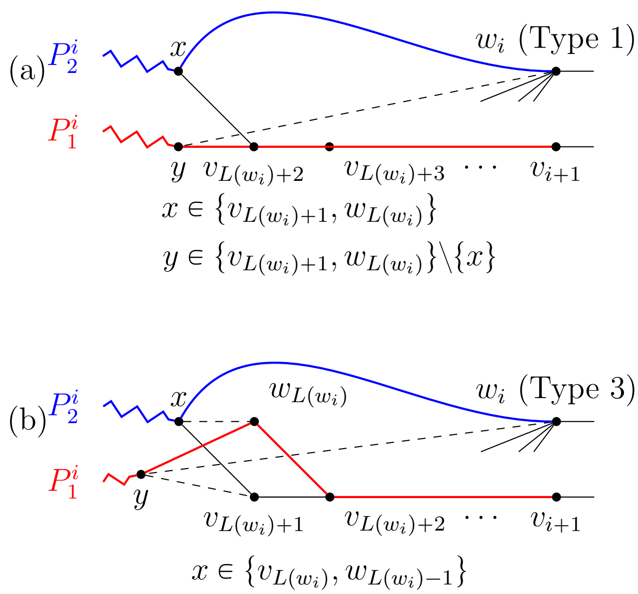


FIGURE 7. Constructing a generalized ladder in a fully patched path. The vertex w_i has Type 1 in (a) and has Type 3 in (b).

adjacent to w_i. We may assume that the paths $P_1^{L(w_i)}$ and $P_2^{L(w_i)}$ end at y and x, respectively. Let P_1^i be a path

$$P_1^{L(w_i)} + yv_{L(w_i)+2}v_{L(w_i)+3} \cdots v_{i+1}$$

and let P_2^i be a path $P_2^{L(w_i)} + xw_i$. See Figure 7. By the induction hypothesis, $V(P_1^{L(w_i)}) \cup V(P_2^{L(w_i)}) \subseteq A_{L(w_i)+1} \subseteq A_{i+1}$, and for each $j \in \{1, 2, \ldots, L(w_i) - 1\}$, $V(P_1^{L(w_i)}) \cup V(P_2^{L(w_i)})$ contains w_j or v_{j+1}. Thus it follows that $V(P_1^i) \cup V(P_2^i) \subseteq A_{i+1}$ and for each $j \in \{1, 2, \ldots, i-1\}$, $V(P_1^i) \cup V(P_2^i)$ contains w_j or v_{j+1}.

We claim that $G[V(P_1^i) \cup V(P_2^i)] + v_{i+1}w_i$ is a generalized ladder with the defining paths P_1^i and P_2^i. By the induction hypothesis, it is enough to show that there are no two crossing chords xa and w_ib for some $a, b \in V(P_1^i)$. Since w_i has no neighbor in $A_{L(w_i)}$ and w_i and y are non-adjacent, $b \in X = \{v_k : k \in \{L(w_i) + 2, L(w_i) + 3, \ldots, i + 1\}\}$. Since x has no neighbor in $X \backslash \{v_{L(w_i)+2}\}$, we deduce that xa and w_ib cannot cross and therefore $G[V(P_1^i) \cup V(P_2^i)] + v_{i+1}w_i$ is a generalized ladder. This proves that if w_i has Type 1, then (P_1^i, P_2^i) is a good pair.

Finally, suppose that w_i has Type 3 and so w_i is adjacent to both $v_{L(w_i)}$ and $w_{L(w_i)-1}$. By symmetry, we may assume that $P_2^{L(w_i)}$ ends at $v_{L(w_i)+1}$. Let x be the predecessor of $v_{L(w_i)+1}$ in $P_2^{L(w_i)}$. Since $P_2^{L(w_i)}$ is on $A_{L(w_i)+1}$ and $v_{L(w_i)+1}$ has only two neighbors $v_{L(w_i)}$, $w_{L(w_i)-1}$ in $A_{L(w_i)+1}$, either $x = v_{L(w_i)}$ or $x = w_{L(w_i)-1}$. Let y be the predecessor of $w_{L(w_i)}$ in $P_1^{L(w_i)}$. Let P_1^i be a path

$$P_1^{L(w_i)} + w_{L(w_i)}v_{L(w_i)+2}v_{L(w_i)+3}\cdots v_{i+1}$$

and let P_2^i be a path obtained from $P_2^{L(w_i)}$ by removing $v_{L(w_i)+1}$ and adding xw_i. See Figure 7(b). It follows from our construction and the induction hypothesis that $V(P_1^i) \cup V(P_2^i) \subseteq A_{i+1}$ and $V(P_1^i) \cup V(P_2^i)$ contains w_j or v_{j+1} for each $j \in \{1, 2, \dots, i-1\}$.

We claim that $G[V(P_1^i) \cup V(P_2^i)] + v_{i+1}w_i$ is a generalized ladder with the defining paths P_1^i and P_2^i. By the induction hypothesis, it is enough to prove that there are no two chords xa and w_ib such that $a, b \in V(P_1^i)$ and b precedes a in P_1^i. Suppose not. Since w_i has no neighbor in $A_{L(w_i)-1}$, neighbors of w_i in P_1^i are in $\{y, w_{L(w_i)}\} \cup \{v_k : k \in \{L(w_i) + 2, L(w_i) + 3, \dots, i + 1\}\}$. Since x has no neighbor in $\{v_k : k \in \{L(w_i) + 2, L(w_i) + 3, \dots, i + 1\}\}$, we deduce that $a = w_{L(w_i)}$ and $b = y$. Since w_i has no neighbor in $A_{L(w_i)-1}$, b is one of $v_{L(w_i)}$ and $w_{L(w_i)-1}$ other than x. Thus $w_{L(w_i)}$ is adjacent to both $v_{L(w_i)}$ and $w_{L(w_i)-1}$. This contradicts (iii) because $v_{L(w_i)+1}$ is also adjacent to both $v_{L(w_i)}$ and $w_{L(w_i)-1}$ and so $G[V(P_1^{L(w_i)}) \cup V(P_2^{L(w_i)})] + v_{L(w_i)+1}w_{L(w_i)}$ is not a generalized ladder. □

Lemma 6.8. *If a graph has a fully patched induced path of length n, then it has a generalized ladder having at least $n + 2$ vertices as an induced subgraph.*

Proof. Let $P = v_0v_1 \cdots v_n$ be the induced path of length n with an $(n-2)$-patch $Q = w_1w_2 \cdots w_{n-2}$. Lemma 6.7 provides a good pair of paths P_1^{n-2} and P_2^{n-2} from $\{v_0, v_1\}$ to $\{v_{n-1}, w_{n-2}\}$ such that $G[V(P_1^{n-2}) \cup V(P_2^{n-2})] + v_{n-1}w_{n-2}$ is a generalized ladder and $V(P_1^{n-2}) \cup V(P_2^{n-2})$ contains w_j or v_{j+1} for each $j \in \{1, 2, \dots, n-3\}$. Since v_n is only adjacent to v_{n-1} and w_{n-2} in G, $G' = G[V(P_1^{n-2}) \cup V(P_2^{n-2}) \cup \{v_n\}]$ is a generalized ladder. Since $v_0, v_1, v_n, v_{n-1}, w_{n-2} \in V(G')$, G' has at least $(n-3) + 5 = n + 2$ vertices. □

Now we are ready to prove the main theorem of this section.

Lemma 6.9. *Let $n \geqslant 1$. If a prime graph has an induced path of length $110592n^7$, then it has a cycle of length $4n + 3$ as a vertex-minor.*

Proof. Let G be a prime graph having an induced path of length $110592n^7$. Suppose that G has no cycle of length $4n+3$ as a vertex-minor. Let $N = 4608n^5$. Then

$$(6(2n)^2 - 2)(N-2) - 1 < 110592n^7.$$

Thus by Proposition 6.5, there exists a graph G' locally equivalent to G having a fully patched induced path of length N. By Lemma 6.8, G' must have a generalized ladder having at least $N+2$ vertices as an induced subgraph. By Proposition 4.1, we deduce that G' has a cycle of length $4n+3$ as a vertex-minor. □

Proof of Theorem 6.1. Let $k = \lfloor n/4 \rfloor$. Let G be a prime graph having a path of length at least $6.75n^7$. Then G has a path of length $6.75(4k)^7 = 110592k^7$, and by Lemma 6.9, G has a cycle of length $4k+3 \geqslant n$ as a vertex-minor. □

7. Main Theorem

In this section, we prove the following.

Theorem 7.1. *For every n, there is N such that every prime graph on at least N vertices has a vertex-minor isomorphic to C_n or $K_n \boxminus K_n$.*

By Theorem 6.1, it is enough to prove the following proposition.

Proposition 7.2. *For every c, there exists N such that every prime graph on at least N vertices has a vertex-minor isomorphic to either P_c or $K_c \boxminus K_c$.*

Here is the proof of Theorem 7.1 assuming Proposition 7.2.

Proof of Theorem 7.1. We take $c = \lceil 6.75n^7 \rceil$ and apply Proposition 7.2 and Theorem 6.1. □

For integers $h, w, \ell \geqslant 1$, a (h, w, ℓ)-*broom* of a graph G is a connected induced subgraph H of G such that

(i) H has an induced path P of length h from some vertex v called the *center*,
(ii) $P \backslash v$ is a component of $H \backslash v$,
(iii) $H \backslash V(P)$ has w components, each having exactly ℓ vertices.

The path P is called a *handle* of H and each component of $H \backslash V(P)$ is called a *fiber* of H. If $H = G$, then we say that G is a (h, w, ℓ)-broom. We call h, w, ℓ the *height*, *width*, *length*, respectively, of a (h, w, ℓ)-broom. See Figure 8. Observe that v has one or more neighbors in each fiber.

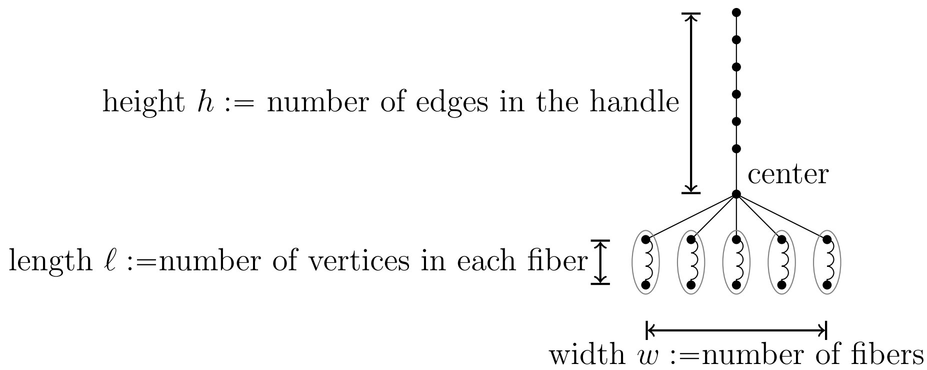


FIGURE 8. A (h, w, ℓ)-broom.

Here is the rough sketch of the proof. If a prime graph G has no vertex-minor isomorphic to P_c or $K_c \boxminus K_c$ and G has a broom having huge width as a vertex-minor, then it has a vertex-minor isomorphic to a broom with larger length and sufficiently big width. So, we increase the length of a broom while keeping its width big. If we obtain a broom of big length by repeatedly applying this process, then we will obtain a broom of larger height. By growing the height, we will eventually obtain a long induced path.

To start the process, we need an initial broom with sufficiently big width. For that purpose, we use the following Ramsey-type theorem.

Theorem 7.3 (folklore; see Diestel [8, Proposition 9.4.1])**.** *For positive integers c and t, there exists $N = g_0(c,t)$ such that every connected graph on at least N vertices must contain K_{t+1}, $K_{1,t}$, or P_c as an induced subgraph.*

By Theorem 7.3, if G is prime and $|V(G)| \geqslant g_0(c, t+1)$, then either G has an induced subgraph isomorphic to P_c or G has a vertex-minor isomorphic to $K_{1,t+1}$. Since a $(1, t, 1)$-broom is isomorphic to $K_{1,t+1}$, we conclude that every sufficiently large prime graph has a vertex-minor isomorphic to a $(1, t, 1)$-broom, unless it has an induced subgraph isomorphic to P_c.

7.1. **Increasing the length of a broom.** We now show that if a prime graph has a broom having sufficiently large width, we can find a broom having larger length after applying local complementation and shrinking the width.

In the following proposition, we want to find a wide broom of length 2 when we are given a sufficiently wide broom of length 1, when the graph has no P_c or $K_c \boxminus K_c$ as a vertex-minor.

Proposition 7.4. *For all integers $c \geqslant 3$ and $t \geqslant 1$, there exists $N = g_1(c,t)$ such that for each $h \geqslant 1$, every prime graph having a $(h,N,1)$-broom has a vertex-minor isomorphic to a $(h,t,2)$-broom, $K_c \boxminus K_c$, or P_c.*

We will use the following theorem.

Theorem 7.5 (Ding, Oporowski, Oxley, Vertigan [10])**.** *For every positive integer n, there exists $N = f(n)$ such that for every bipartite graph G with a bipartition (S,T), if no two vertices in S have the same set of neighbors and $|S| \geqslant N$, then S and T have n-element subsets S' and T', respectively, such that $G[S',T']$ is isomorphic to $\overline{K_n} \boxminus \overline{K_n}$, $\overline{K_n} \boxslash \overline{K_n}$, or $\overline{K_n} \boxtimes \overline{K_n}$.*

Proof of Proposition 7.4. Let $N = f(R(w,w))$ where f is the function in Theorem 7.5, and $w = \max(t+(c-1)(c-3), 2c-1)$. Suppose that G has a $(h, g_1(c,t), 1)$-broom H. Note that every fiber of H is a single vertex.

Let S be the union of the vertex sets of all fibers of H, and x be the center of H. Let $N_G(S) \setminus \{x\} = T$. Since G is prime, no two vertices in G have the same set of neighbors, and so two distinct vertices in S have different sets of neighbors in T. Since $|S| = N = f(R(w,w))$, by Theorem 7.5, there exist $S_0 \subseteq S$, $T_0 \subseteq T$ such that $G[S_0,T_0]$ is isomorphic to $\overline{K_{R(w,w)}} \boxminus \overline{K_{R(w,w)}}$, $\overline{K_{R(w,w)}} \boxslash \overline{K_{R(w,w)}}$ or $\overline{K_{R(w,w)}} \boxtimes \overline{K_{R(w,w)}}$. Since $|T_0| \geqslant R(w,w)$, by Ramsey's theorem, there exist $S' \subseteq S_0$ and $T' \subseteq T_0$ such that $G[S',T']$ is isomorphic to $\overline{K_w} \boxminus \overline{K_w}$, $\overline{K_w} \boxslash \overline{K_w}$, or $\overline{K_w} \boxtimes \overline{K_w}$, and T' is a clique or a stable set in G. If $G[S',T']$ is isomorphic to $\overline{K_w} \boxslash \overline{K_w}$ or $\overline{K_w} \boxtimes \overline{K_w}$, then by Lemmas 2.7 and 2.8, G has a vertex-minor isomorphic to either P_{2w} or $K_{w-2} \boxminus K_{w-2}$. Since $w \geqslant 2c-1$ and $c \geqslant 3$, we have P_c or $K_c \boxminus K_c$. Thus we may assume that $G[S',T']$ is isomorphic to $\overline{K_w} \boxminus \overline{K_w}$.

If T' is a clique in G, then we can remove the edges connecting T' with x by applying local complementation at some vertices in S'. Thus, we can obtain a vertex-minor isomorphic to $K_w \boxminus K_w$ from $G[S' \cup T' \cup \{x\}]$ by applying local complementation at x and deleting x. Therefore we may assume that T' is a stable set in G.

We claim that each vertex $y \neq x$ in the handle of H is adjacent to at most c vertices in T', or G has $K_c \boxminus K_c$ as a vertex-minor. Suppose not. If y is a neighbor of x, then by pivoting an edge of $G[S',T']$, we can delete the edge xy. From there, we obtain a vertex-minor isomorphic to $K_c \boxminus K_c$ by applying local complementation at x and y. If y is not adjacent to x, then we obtain a vertex-minor isomorphic to $K_c \boxminus K_c$

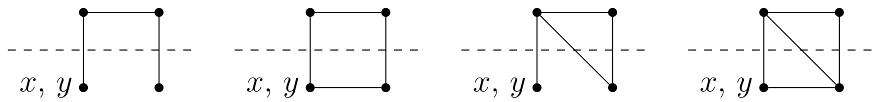


FIGURE 9. Dealing with 4-vertex graphs in Lemma 7.6.

by deleting all vertices in the handle other than x and y, and applying local complementation at x and y. This proves the claim.

By deleting at most $(c-1)h$ vertices in T' and their pairs in S', we can assume that no vertex other than x in the handle has a neighbor in T' and this broom has width at least $w-(c-1)h$. If $h+2 \geqslant c$, then we have P_c as an induced subgraph. Thus we may assume that $h \leqslant c-3$. Since $w-(c-1)h \geqslant w-(c-1)(c-3) \geqslant t$, we obtain a vertex-minor isomorphic to a $(h,t,2)$-broom. □

We now aim to increase the length of a broom when the broom has length at least 2. For a fiber F of a broom H, we say that a vertex $v \in V(G)\backslash V(H)$ *blocks* F if

$$\rho^*_G(V(F), (V(H)\backslash V(F)) \cup \{v\}) > 1.$$

If G is prime and F has at least two vertices, then G has a blocking sequence for $(V(F), V(H)\backslash V(F))$ by Proposition 5.2 and therefore there exists a vertex v that blocks F because we can take the first vertex in the blocking sequence.

Lemma 7.6. *Let G be a graph and let x, y be two vertices such that $\rho_G(\{x,y\}) = 2$ and $G\backslash x\backslash y$ is connected. Then there exists some sequence $v_1, v_2, \ldots, v_n \in V(G)\backslash\{x,y\}$ of (not necessarily distinct) vertices such that $G * v_1 * v_2 \cdots * v_n$ has an induced path of length 3 from x to y.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| = 4$, then it is easy to check all cases to obtain a path of length 3. To do so, first observe that up to symmetry, there are 2 cases in $G[\{x,y\}, V(G)\backslash\{x,y\}]$; either it is a matching of size 2 or a path of length 3. In both cases, one can find a desired sequence of vertices to apply local complementation, see Figure 9 for all possible graphs on 4-vertices up to isomorphism.

Now we may assume that G has at least 5 vertices. Let $A_1 = N_G(x)\backslash(N_G(y) \cup \{y\})$, $A_2 = N_G(x) \cap N_G(y)$, and $A_3 = N_G(y)\backslash(N_G(x) \cup \{x\})$. Clearly $\rho_G(\{x,y\}) = 2$ is equivalent to say that at least two of A_1, A_2, A_3 are nonempty.

We say a vertex t in $G\backslash x\backslash y$ *deletable* if $G\backslash x\backslash y\backslash t$ is connected. If there is a deletable vertex not in $A_1 \cup A_2 \cup A_3$, then $\rho_{G\backslash t}(\{x,y\}) = 2$ and we apply the induction hypothesis to find an induced path. Thus we may assume that all deletable vertices are in $A_1 \cup A_2 \cup A_3$.

If $|A_i| > 1$ and A_i has a deletable vertex t for some $i = 1, 2, 3$, then $\rho_{G\backslash t}(\{x, y\}) = 2$ and so we obtain a sequence by applying the induction hypothesis. So we may assume that if A_i has a deletable vertex, then $|A_i| = 1$.

If there are three deletable vertices t_1, t_2, t_3 in $G\backslash x\backslash y$, then we may assume $A_i = \{t_i\}$. However, $\rho_{G\backslash t_1}(\{x, y\}) = 2$ because A_2, A_3 are nonempty and therefore we obtain an induced path from x to y by the induction hypothesis.

Thus we may assume that $G\backslash x\backslash y$ has at most 2 deletable vertices. So $G\backslash x\backslash y$ has maximum degree at most 2 because otherwise we can choose leaves of a spanning tree of $G\backslash x\backslash y$ using all edges incident with a vertex of the maximum degree. If $G\backslash x\backslash y$ is a cycle, then every vertex is deletable and so $G\backslash x\backslash y$ is a path. Let w be a degree-2 vertex in $G\backslash x\backslash y$. Then $G * w$ has at least 3 deletable vertices and therefore we find a desired sequence $v_1, v_2, \ldots, v_n$ such that $G * w * v_1 * v_2 \cdots * v_n$ has an induced path of length 3 from x to y. □

Lemma 7.7. *Let G be a graph and let x, y be two vertices in G, and let $F_1, F_2, \ldots, F_c$ be the components of $G\backslash x\backslash y$. If $\rho^*_G(\{x, y\}, F_i) = 2$ for all $1 \leqslant i \leqslant c$, then G has a vertex-minor isomorphic to $K_c \boxminus K_c$.*

Proof. We proceed by induction on $|V(G)| + |E(G)|$.

Suppose that $G[V(F_i) \cup \{x, y\}]$ is not an induced path of length 3 from x to y. By Lemma 7.6, there exists a sequence $v_1, v_2, \ldots, v_n \in V(F_i)$ such that $G[V(F_i) \cup \{x, y\}] * v_1 * v_2 \cdots * v_n$ has an induced path of length 3 from x to y. If $|V(F_i)| \geqslant 3$, then we delete all vertices in F_i not on this path and apply the induction hypothesis. If $|V(F_i)| = 2$, then $|E(G[V(F_i) \cup \{x, y\}])| > |E(G[V(F_i) \cup \{x, y\}] * v_1 * v_2 * \cdots * v_n)|$ because two vertices in F_i are connected, $G[\{x, y\}, V(F_i)]$ has at least two edges, and $G[V(F_i) \cup \{x, y\}]$ is not an induced path of length 3 from x to y. So we apply the induction hypothesis to $G * v_1 * v_2 * \cdots * v_n$ to obtain a vertex-minor isomorphic to $K_c \boxminus K_c$.

Therefore we may assume that $G[V(F_i) \cup \{x, y\}]$ is an induced path of length 3 from x to y for all i. Thus $G * x * y\backslash x\backslash y$ is indeed isomorphic to $K_c \boxminus K_c$. □

Lemma 7.8. *Let t be a positive integer, and G be a bipartite graph with a bipartition (S, T) such that every vertex in T has degree at least 1. Then either S has a vertex of degree at least $t + 1$ or G has an induced matching of size at least $|T|/t$.*

Proof. We claim that if every vertex in S has degree at most t, then G has an induced matching of size at least $|T|/t$. We proceed by induction on $|T|$. This is trivial if $|T| = 0$. If $0 < |T| \leqslant t$, then we can simply pick an edge to form an induced matching of size 1. So we may assume that $|T| > t$.

We may assume that T has a vertex w of degree 1, because otherwise we can delete a vertex in S and apply the induction hypothesis. Let v be the unique neighbor of w. By the induction hypothesis, $G \backslash v \backslash N_G(v)$ has an induced matching M' of size at least $(|T|-t)/t$. Now $M' \cup \{vw\}$ is a desired induced matching. □

Lemma 7.9. *Let H be a broom in a graph G having n fibers $F_1, F_2, \ldots, F_n$ given with n vertices $v_1, v_2, \ldots, v_n$ in $V(G) \backslash V(H)$ such that*

(1) v_i blocks F_j if and only if $i = j$,
(2) v_i has a neighbor in F_j if and only if $i \leqslant j$.

If $n \geqslant R(c+1, c+1)$, then G has a vertex-minor isomorphic to P_c.

Proof. If $j > i$, then v_i has a neighbor in F_j, but v_i does not block F_j. Therefore, v_i is adjacent to every vertex in $V(F_j) \cap N_H(x)$ for $j > i$. Since $n \geqslant R(c+1, c+1)$, there exist $1 \leqslant t_1 < t_2 \cdots < t_{c+1} \leqslant n$ such that $\{v_{t_1}, v_{t_2}, \ldots, v_{t_{c+1}}\}$ is a clique or a stable set of G. For $1 \leqslant i \leqslant c+1$, let w_i be a vertex in $V(F_{t_i}) \cap N_H(x)$. Clearly,

$$G[\{v_{t_1}, v_{t_3}, \ldots, v_{t_{2\lceil c/2\rceil-1}}\}, \{w_2, w_4, \ldots, w_{2\lceil c/2\rceil}\}]$$

is isomorphic to $\overline{K_{\lceil c/2\rceil}} \boxslash \overline{K_{\lceil c/2\rceil}}$.

By Lemma 2.8, $\overline{K_{\lceil c/2\rceil}} \boxslash \overline{K_{\lceil c/2\rceil}}$ or $\overline{K_{\lceil c/2\rceil}} \boxslash K_{\lceil c/2\rceil}$ has a vertex-minor isomorphic to P_c. □

Lemma 7.10. *Let H be a broom in a graph G having n fibers $F_1, F_2, \ldots, F_n$. Let $v_1, v_2, \ldots, v_n$ be vertices in $V(G) \backslash V(H)$ such that*

(1) v_i blocks F_j if and only if $i = j$,
(2) v_i has a neighbor in F_j for all i and j.

If $n \geqslant R(c+2, c+2)$, then G has a vertex-minor isomorphic to $K_c \boxminus K_c$.

Proof. If $i \neq j$, then v_j does not block F_i and therefore $N_G(v_j) \cap V(F_i) = N_G(x) \cap V(F_i)$. Since $n \geqslant R(c+2, c+2)$, there exist $1 \leqslant t_1 < t_2 \cdots < t_{c+2} \leqslant n$ such that $\{v_{t_1}, v_{t_2}, \ldots, v_{t_{c+2}}\}$ is a clique or a stable set of G.

We claim that for each $1 \leqslant i \leqslant c+2$, there exist a sequence $w_1^{(i)}, w_2^{(i)}, \ldots, w_{k_i}^{(i)}$ of $k_i \geqslant 0$ vertices in $V(F_{t_i}) \backslash (N_G(x) \cup N_G(v_{t_i}))$ and $z_i \in V(F_{t_i})$ such that z_i is not adjacent to v_{t_i} in $G * w_1^{(i)} * w_2^{(i)} * \cdots * w_{k_i}^{(i)}$ but z_i is adjacent to v_{t_j} in $G * w_1^{(i)} * w_2^{(i)} * \cdots * w_{k_i}^{(i)}$ for all $j \neq i$.

Let $A_1^{(i)} = (N_G(v_{t_i}) \backslash N_G(x)) \cap V(F_{t_i})$, $A_2^{(i)} = (N_G(v_{t_i}) \cap N_G(x)) \cap V(F_{t_i})$ and $A_3^{(i)} = (N_G(x) \backslash N_G(v_{t_i})) \cap V(F_{t_i})$.

If $A_3^{(i)} \neq \emptyset$, then a vertex z_i in $A_3^{(i)}$ satisfies the claim. So we may assume $A_3^{(i)}$ is empty. Then $A_1^{(i)} \neq \emptyset$ and $A_2^{(i)} \neq \emptyset$, otherwise $\rho_G^*(\{v_{t_i}, v_{t_j}\}, V(F_{t_i})) \leqslant 1$ for all $j \neq i$ because $N_G(v_{t_j}) \cap V(F_{t_i}) = N_G(x) \cap V(F_{t_i})$. We choose $a_1^{(i)} \in A_1^{(i)}$ and $a_2^{(i)} \in A_2^{(i)}$ so that the distance from $a_1^{(i)}$ to $a_2^{(i)}$ in F_i is minimum.

Let P_i be a shortest path from $a_1^{(i)}$ to $a_2^{(i)}$ in F_{t_i}. Note that each internal vertex of P_i is not contained in $A_1^{(i)} \cup A_2^{(i)}$. After applying local complementation at all internal vertices of P_i, $a_1^{(i)}$ is adjacent to $a_2^{(i)}$ and v_{t_i}, and non-adjacent to v_{t_j} for all $j \neq i$. So by applying one more local complementation at $a_1^{(i)}$ if necessary, we can delete the edges between $a_2^{(i)}$ and v_{t_j} for all $j \neq i$. And then, $z_i = a_2^{(i)}$ satisfies the claim.

Now, take $G' = G * w_1^{(1)} * \cdots * w_{k_1}^{(1)} * w_1^{(2)} * \cdots * w_{k_2}^{(2)} \cdots * w_1^{(c+2)} * \cdots * w_{k_{c+2}}^{(c+2)}$. Since each $w_k^{(i)}$ has no neighbors in $\{v_{t_1}, v_{t_2}, \ldots, v_{t_{c+2}}\}$ in G, applying local complementation at $w_k^{(i)}$ does not change the adjacency between any two vertices in $\{v_{t_1}, v_{t_2}, \ldots, v_{t_{c+2}}\}$. Thus the induced subgraph of G' on $\{z_1, z_2, \ldots, z_{c+2}\} \cup \{v_{t_1}, v_{t_2}, \ldots, v_{t_{c+2}}\}$ is isomorphic to $\overline{K_{c+2}} \boxtimes \overline{K_{c+2}}$ or $\overline{K_{c+2}} \boxtimes K_{c+2}$, and by Lemma 2.7, G has a vertex-minor isomorphic to $K_c \boxminus K_c$. □

Lemma 7.11. *Let H be a (h, n, ℓ)-broom in a graph G having n fibers $F_1, F_2, \ldots, F_n$ given with n vertices $v_1, v_2, \ldots, v_n$ in $V(G) \backslash V(H)$ such that*

(1) v_i blocks F_j if and only if $i = j$,
(2) if $i \neq j$, then v_i has no neighbor in F_j.

If $n \geqslant R(t + (c-1)(c-3), c)$, then G has a vertex-minor isomorphic to P_c, $K_c \boxminus K_c$, or a $(h, t, \ell + 1)$-broom.

Proof. Since $n \geqslant R(t + (c-1)(c-3), c)$, there exist $1 \leqslant t_1 < t_2 \cdots < t_k \leqslant n$ such that either

(1) $k = c$ and $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$ is a clique in G, or
(2) $k = t + (c-1)(c-3)$ and $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$ is a stable set in G.

First, we assume that $k = c$ and $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$ is a clique. For each t_i, since $\rho_G^*(\{x, v_{t_i}\}, V(F_{t_i})) \geqslant 2$, by Lemma 7.6, there exists some sequence $w_1, w_2, \ldots, w_n \in V(F_{t_i})$ of (not necessarily distinct) vertices such that $G[V(F_{t_i}) \cup \{x, v_{t_i}\}] * w_1 * w_2 \cdots * w_n$ has an induced path of length 2 from v_{t_i} to x. By applying local complementation at x, we have a vertex-minor isomorphic to $K_c \boxminus K_c$.

Now, suppose that $k = t + (c-1)(c-3)$ and $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$ is a stable set in G. Let P be the handle of H. If $h + 2 \geqslant c$, then we have P_c as an induced subgraph. Thus we may assume that $h \leqslant c-3$. We assume that a vertex $y \in V(P)\backslash\{x\}$ adjacent to c vertices in $\{v_1, v_2, \ldots, v_k\}$. Then since $\rho^*_G(\{x,y\}), V(F_i) \cup \{v_{t_i}\}) = 2$ for each i, by Lemma 7.7, we have a vertex-minor isomorphic to $K_c \boxminus K_c$. Thus, every vertex in the handle other than x cannot have more than $c-1$ neighbors in $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$. By deleting at most $(c-1)h$ vertices in $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$, we can remove all edges from $V(P)\backslash\{x\}$ to $\{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$. Since

$$k-(c-1)h \geqslant k-(c-1)(c-3) \geqslant t,$$

we have a vertex-minor isomorphic to a $(h, t, \ell+1)$-broom. □

Proposition 7.12. *For positive integers c and t, there exists $N = g_2(c,t)$ such that for all integers $\ell \geqslant 2$ and $h \geqslant 1$, every prime graph having a (h, N, ℓ)-broom has a vertex-minor isomorphic to a $(h, t, \ell+1)$-broom, P_c, or $K_c \boxminus K_c$.*

Proof. Let $N = g_2(c,t) = (c-1)m$, where $m = R(m_1, m_2, m_2, m_2)$, $m_1 = R(t+(c-1)(c-3), c)$, and $m_2 = R(c+2, c+2)$. Let H be a (h, N, ℓ)-broom of G. If a vertex w in $V(G)\backslash V(H)$ blocks c fibers of H, then for each fiber F of them, $\rho^*_G(\{w,x\}, V(F)) = 2$. So by Lemma 7.7, G has a vertex-minor isomorphic to $K_c \boxminus K_c$. Thus, a vertex in $V(G)\backslash V(H)$ can block at most $c-1$ fibers of H.

For each fiber F of H, there is a vertex $v \in V(G)\backslash V(H)$ that blocks F because G is prime. Thus, by Lemma 7.8, there are $g_2(c,t)/(c-1) = m$ vertices $v_1, v_2, \ldots, v_m$ in $V(G)\backslash V(H)$ and fibers $F_1, F_2, \ldots, F_m$ of H such that for $1 \leqslant i,j \leqslant m$, v_i blocks F_j if and only if $i=j$. For $i \neq j$, either v_i has no neighbor in F_j or v_i has a neighbor in F_j but $\rho^*_G(\{v_i, x\}, V(F_j)) = 1$.

We assume that $V(K_m) = \{1, 2, \ldots, m\}$. We color the edges of K_m such that an edge $\{i,j\}$ is

- green if $N_G(v_i) \cap V(F_j) \neq \varnothing$ and $N_G(v_j) \cap V(F_i) \neq \varnothing$,
- red if $N_G(v_i) \cap V(F_j) \neq \varnothing$ and $N_G(v_j) \cap V(F_i) = \varnothing$,
- yellow if $N_G(v_i) \cap V(F_j) = \varnothing$ and $N_G(v_j) \cap V(F_i) \neq \varnothing$,
- blue if $N_G(v_i) \cap V(F_j) = N_G(v_j) \cap V(F_i) = \varnothing$.

Since $|V(K_m)| = m = R(m_1, m_2, m_2, m_2)$, by Ramsey's theorem, either K_m has a green clique of size m_1, or K_m has a monochromatic clique of size m_2 which is red, yellow, or blue.

If K_m has a red clique C of size m_2, then for $i,j \in C$, v_i has a neighbor in F_j if and only if $i \leqslant j$. Since $m_2 \geqslant R(c+1, c+1)$, by Lemma 7.9, G has a vertex-minor isomorphic to P_c.

Similarly, if K_m has a yellow clique C of size m_2, by Lemma 7.9, G has a vertex-minor isomorphic to P_c.

If K_m has a blue clique C of size m_2, then for distinct $i, j \in C$, v_i has a neighbor in F_j. Since $m_2 = R(c+2, c+2)$, by Lemma 7.10, G has a vertex-minor isomorphic to $K_c \boxminus K_c$.

If K_m has a green clique C of size m_1, then for distinct $i, j \in C$, v_i has no neighbor in F_j. Since $m_1 = R(t + (c-1)(c-3), c)$, by Lemma 7.11, G has a vertex-minor isomorphic to P_c, $K_c \boxminus K_c$, or a $(h, t, \ell+1)$-broom. $\square$

7.2. **Increasing the height of a broom.**

Proposition 7.13. *For positive integers c, t, there exists $N = g_3(c,t)$ such that for $h \geqslant 1$, every prime graph having a $(h, 1, N)$-broom has a vertex-minor isomorphic to a $(h+1, t, 1)$-broom or P_c.*

Proof. Let $N = g_3(c,t) = g_0(c, 2t)$ where g_0 is given in Theorem 7.3. Suppose that G has a $(h, 1, N)$-broom H and let x be the center of H. Let F be the fiber of H.

Since F is connected, by Theorem 7.3, F has an induced subgraph isomorphic to P_c, or F has a vertex-minor isomorphic to K_{2t+1}. We may assume that F has an induced subgraph F' isomorphic to K_{2t+1}. Let $P = p_1 p_2 \ldots p_m$ be a shortest path from $p_1 = x$ to F' in H. Note that $m \geqslant 2$ and p_{m-1} is adjacent to at least one vertices of F'. Let $S = N_H(p_{m-1}) \cap V(F')$.

We claim that there exists a vertex $v \in V(F')$ such that $(G * v)[V(F) \cup \{x\}]$ has an induced path of length at least $m-1$ from x, and the last vertex of the path has t neighbors in F' which form a stable set in G.

If $|S| \leqslant t$, then choose $p_{m+1} \in V(F') \backslash S$ and we delete $S \backslash p_m$ from F'. And by applying local complementation at p_{m+1}, we obtain a path from x to p_{m+1} such that p_{m+1} has t neighbors in F' which form a stable set.

If $|S| \geqslant t+1$, then by applying local complementation at p_m, we obtain a path from x to p_m such that p_m has t neighbors in F' which form a stable set. Thus, we prove the claim.

Since $m \geqslant 2$, the union of the handle of H and the path in the claim form a path of length at least $h+1$, and the last vertex of the path has t neighbors which form a stable set in F'. Therefore, G has a vertex-minor isomorphic to a $(h+1, t, 1)$-broom. $\square$

Proposition 7.14. *For positive integers c, t, there exists $N = g_4(c,t)$ such that for all $h \geqslant 1$, every prime graph having a $(h, N, 1)$-broom has a vertex-minor isomorphic to a $(h+1, t, 1)$-broom, P_c, or $K_c \boxminus K_c$.*

Proof. By Proposition 7.13, there exists N_0 depending only on c and t such that every prime graph having a $(h,1,N_0)$-broom has a vertex-minor isomorphic to a $(h+1,t,1)$-broom or P_c. By applying Proposition 7.12 (N_0-2) times, we deduce that there exists N_1 such that every prime graph having a $(h,N_1,2)$-broom has a vertex-minor isomorphic to a $(h,1,N_0)$-broom, P_c, or $K_c \boxminus K_c$. By Proposition 7.4, there exists N such that every prime graph having a $(h,N,1)$-broom has a vertex-minor isomorphic to a $(h,N_1,2)$-broom, P_c, or $K_c \boxminus K_c$. □

We are now ready with all necessary lemmas to prove Proposition 7.2.

Proof of Proposition 7.2. By Theorem 1.1, every prime graph on at least 5 vertices has a vertex-minor isomorphic to C_5 and P_4 is a vertex-minor of C_5. Therefore we may assume that $c \geqslant 5$.

By applying Proposition 7.14 $(c-3)$ times, we deduce that there exists a big integer t depending only on c such that every prime graph G with a $(1,t,1)$-broom has a vertex-minor isomorphic to a $(c-2,1,1)$-broom, P_c, or $K_c \boxminus K_c$. Since a $(c-2,1,1)$-broom is isomorphic to P_c and a $(1,t,1)$-broom is isomorphic to $K_{1,t+1}$, we conclude that every prime graph having a vertex-minor isomorphic to $K_{1,t+1}$ has a vertex-minor isomorphic to P_c or $K_c \boxminus K_c$. By Theorem 3.1, there exists N such that every connected graph on at least N vertices has a vertex-minor isomorphic to $K_{1,t+1}$. This completes the proof. □

8. Why optimal?

Our main theorem (Theorem 7.1) states that sufficiently large prime graphs must have a vertex-minor isomorphic to C_n or $K_n \boxminus K_n$. But do we really need these two graphs? To justify why we need both, we should show that for some n, C_n is not a vertex-minor of $K_N \boxminus K_N$ for all N and similarly $K_n \boxminus K_n$ is not a vertex-minor of C_N for all N, because C_n and $K_n \boxminus K_n$ are also prime.

Proposition 8.1. *Let n be a positive integer.*

(1) $K_3 \boxminus K_3$ is not a vertex-minor of C_n.
(2) C_7 is not a vertex-minor of $K_n \boxminus K_n$.

Since C_7 is a vertex-minor of C_n for all $n \geqslant 7$, the above proposition implies that C_n is not a vertex-minor of $K_N \boxminus K_N$ when $n \geqslant 7$. Similarly $K_n \boxminus K_n$ is not a vertex-minor of C_N for all $n \geqslant 3$.

We can classify all non-trivial prime vertex-minors of a cycle graph.

Lemma 8.2. *If a prime graph H on at least 5 vertices is a vertex-minor of C_n, then H is locally equivalent to a cycle graph.*

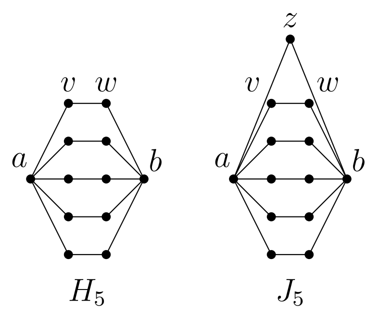


FIGURE 10. The graphs H_5 and J_5.

Proof. We proceed by induction on n. If $n = 5$, then it is trivial. Let us assume $n > 5$. Suppose $|V(H)| < |V(C_n)|$. By Lemma 2.1, H is a vertex-minor of $C_n \backslash v$, $C_n * v \backslash v$, or $C_n \wedge vw \backslash v$ for a neighbor w of v.

If H is vertex-minor of $C_n * v \backslash v$, then we can apply the induction hypothesis because $C_n * v \backslash v$ is isomorphic to C_{n-1}.

By Lemma 2.6, H cannot be a vertex-minor of $C_n \backslash v$ because $C_n \backslash v$ has no prime induced subgraph on at least 5 vertices.

Thus we may assume that H is a vertex-minor of $C_n \wedge vw \backslash v$ for a neighbor w of v. Again, by Lemma 2.6, H is isomorphic to a vertex-minor of C_{n-2}. □

Classifying prime vertex-minors of $K_n \boxminus K_n$ turns out to be more tedious. Instead of identifying prime vertex-minors of $K_n \boxminus K_n$, we focus on characterizing prime vertex-minors on 7 vertices to prove (2) of Proposition 8.1.

Instead of $K_n \boxminus K_n$, we will first consider H_n. Let H_n be the graph having two specified vertices called *roots* and n internally disjoint paths of length 3 joining the roots. Let J_n be the graph obtained from H_n by adding a common neighbor of two roots. Then H_n has $2n + 2$ vertices and J_n has $2n + 3$ vertices, see Figure 10. It is easy to observe the following.

Lemma 8.3. *Let H be a prime vertex-minor of H_n on at least 5 vertices. If $|V(H_n)| - |V(H)| \geqslant 3$, then J_{n-1} has a vertex-minor isomorphic to H.*

Proof. We may assume $n \geqslant 3$. Since at most 2 vertices of H_n have degree other than 2, there exists $v \in V(H_n) \backslash V(H)$ of degree 2 in H_n. Let w be the neighbor of v having degree 2 in H_n. Let $av'w'b$ be a path of length 3 from a to b in H_n such that $\{v, w\} \neq \{v', w'\}$.

By Lemma 2.1, H is a vertex-minor of either $H_n\backslash v$, $H_n * v\backslash v$ or $H_n \wedge vw\backslash v$. If H is a vertex-minor of $H_n * v\backslash v$, then H is isomorphic to a vertex-minor of J_{n-1}, because $H_n * v\backslash v$ is isomorphic to J_{n-1}.

Since w has degree 1 in $H_n\backslash v$, by Lemma 2.6, if H is a vertex-minor of $H_n\backslash v$, then H is isomorphic to a vertex-minor of $H_n\backslash v\backslash w$. Since $H_n\backslash v\backslash w$ is isomorphic to H_{n-1} and H_{n-1} is an induced subgraph of J_{n-1}, H is isomorphic to a vertex-minor of J_{n-1}.

Similarly, if H is a vertex-minor of $H_n \wedge vw\backslash v$, then H is isomorphic to a vertex-minor of $H_n \wedge vw\backslash v\backslash w$. Clearly, $(H_n \wedge vw\backslash v\backslash w) \wedge v'w'$ is isomorphic to H_{n-1}. Since H_{n-1} is an induced subgraph of J_{n-1}, H is isomorphic to a vertex-minor of J_{n-1}, as required. □

Lemma 8.4. *Let H be a prime vertex-minor of J_n on at least 5 vertices. If $|V(J_n)| - |V(H)| \geqslant 2$, then H_n has a vertex-minor isomorphic to H.*

Proof. We may assume $n \geqslant 2$. Let a, b be the roots of J_n, azb be the path of length 2, and $avwb$ be a path of length 3 from a to b.

Case 1: Suppose that $V(J_n)\backslash V(H)$ has a degree-2 vertex on a path of length 3 from a to b. We may assume that it is v by symmetry. By Lemma 2.1, H is a vertex-minor of $J_n\backslash v$, $J_n * v\backslash v$, or $J_n \wedge vw\backslash v$.

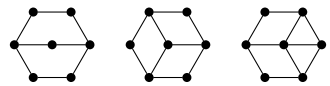
If H is a vertex-minor of $J_n\backslash v$, then H is isomorphic to a vertex-minor of $J_n\backslash v\backslash w$ by Lemma 2.6, because w has degree 1 in $J_n\backslash v$. Similarly, if H is a vertex-minor of $J_n \wedge vw\backslash v$, then H is isomorphic to a vertex-minor of $J_n \wedge vw\backslash v\backslash w$. Clearly, $J_n\backslash v\backslash w$ and $(J_n \wedge vw\backslash v\backslash w) * z$ are isomorphic to J_{n-1}, and J_{n-1} is a vertex-minor of H_n.

If H is a vertex-minor of $J_n * v\backslash v$, then by Lemma 2.6, H is isomorphic to a vertex-minor of $J_n * v\backslash v\backslash w$, which is isomorphic to J_{n-1}, because w and z have the same set of neighbors in $J_n * v\backslash v$. Since J_{n-1} is a vertex-minor of H_n, H is isomorphic to a vertex-minor of H_n. This proves the lemma in Case 1.

Case 2: Suppose that $z \in V(J_n)\backslash V(H)$. Then by Lemma 2.1, H is a vertex-minor of $J_n\backslash z$, $J_n * z\backslash z$, or $J_n \wedge az\backslash z$. Since $J_n\backslash z$ and $(J_n * z\backslash z) \wedge vw$ are isomorphic to H_n, we may assume that H is a vertex-minor of $J_n \wedge az\backslash z$. However, $J_n \wedge az\backslash z$ has no prime induced subgraph on at least 5 vertices and therefore by Lemma 2.6, H cannot be a vertex-minor of $J_n \wedge az\backslash z$, contradicting our assumption.

Case 3: Suppose that a or b is contained in $V(J_n)\backslash V(H)$. By symmetry, let us assume $a \in V(J_n)\backslash V(H)$. By Lemma 2.1, H is a vertex-minor of $J_n\backslash a$, $J_n * a\backslash a$, or $J_n \wedge az\backslash a$.

Since $J_n\backslash a$ has no prime induced subgraph on at least 5 vertices, H cannot be a vertex-minor of $J_n\backslash a$ by Lemma 2.6.



F_1 F_2 F_3

FIGURE 11. Graphs F_1, F_2 and F_3.

Suppose H is a vertex-minor of $J_n \wedge az \backslash a$. By the definition of pivoting, b is adjacent to all vertices of $N_{J_n}(a) \backslash \{z\}$ in $J_n \wedge az \backslash a$. We can remove all these edges between b and $N_{J_n}(a) \backslash \{z\}$ by applying local complementation on all vertices of $N_{J_n}(b) \backslash \{z\}$ in $J_n \wedge az \backslash a$. Thus, H_n is locally equivalent to $J_n \wedge az \backslash a$, and H is isomorphic to a vertex-minor of H_n.

Now suppose that H is a vertex-minor of $J_n * a \backslash a$. By the definition of local complementation, $N_{J_n}(a)$ forms a clique in $J_n * a \backslash a$. So, b is adjacent to all vertices of $N_{J_n}(a) \backslash \{z\}$ in $(J_n * a \backslash a) * z$. Similarly in the above case, by applying local complementation on all vertices of $N_{J_n}(b) \backslash \{z\}$ in $(J_n * a \backslash a) * z$, we can remove all edges between b and $N_{J_n}(a) \backslash \{z\}$ in $(J_n * a \backslash a) * z$. Finally, by pivoting vw, we can remove the edge bz, and therefore, $J_n * a \backslash a$ is locally equivalent to H_n. Thus, H is isomorphic to a vertex-minor of H_n. □

Let F_1, F_2, F_3 be the graphs in Figure 11.

Lemma 8.5. *Let $n \geqslant 3$ be an integer. If a prime graph H is a vertex-minor of H_n and $|V(H)| = 7$, then H is locally equivalent to F_1, F_2, or F_3.*

Proof. We proceed by induction on n. If $n = 3$, then let H be a prime 7-vertex vertex-minor of H_3. Let $axyb$ be a path from a root a to the other root b in H_3. By symmetry, we may assume that $V(H_3) \backslash V(H) = \{x\}$ or $\{a\}$. By Lemma 2.1, H is locally equivalent to $H_3 \backslash x$, $H_3 * x \backslash x$, $H_3 \wedge xa \backslash x$, $H_3 \backslash a$, $H_3 * a \backslash a$, or $H_3 \wedge ab \backslash a$. The conclusion follows because $H_3 \backslash x$, $H_3 \wedge xy \backslash x$, $H_3 \backslash a$ are not prime and $H_3 * x \backslash x$, $H_3 \wedge ax \backslash a$, and $H_3 * a \backslash a$ are isomorphic to F_1, F_2, and F_3, respectively.

Suppose $n > 3$. By Lemma 8.3, every 7-vertex prime vertex-minor is also isomorphic to a vertex-minor of J_{n-1}. By Lemma 8.4, it is isomorphic to a vertex-minor of H_{n-1}. The conclusion follows from the induction hypothesis. □

Lemma 8.6. *The graphs F_1, F_2, F_3 are not locally equivalent to C_7.*

Proof. Suppose that F_i is locally equivalent to C_7. Then $\rho_{F_i}(X) = \rho_{C_7}(X)$ for all $X \subseteq V(C_7)$ by Lemma 2.3. Let x be the vertex in the

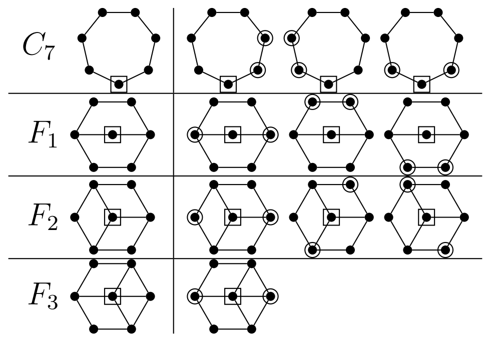


FIGURE 12. List of all 3-vertex sets having cut-rank 2 containing a fixed vertex x denoted by a square.

center of F_i, see Figure 12. By symmetry of C_7, we may assume that x is mapped to a particular vertex in C_7. Figure 12 presents all vertex subsets of size 3 having cut-rank 2 and containing x in graphs C_7, F_1, F_2, F_3. It is now easy to deduce that no bijection on the vertex set will map these subsets correctly. □

We are now ready to prove Proposition 8.1.

Proof of Proposition 8.1. (1) By Lemma 8.2, it is enough to check that $K_3 \boxminus K_3$ is not locally equivalent to C_6. This can be checked easily.

(2) By applying local complementation at roots, we can easily see that H_n has a vertex-minor isomorphic to $K_n \boxminus K_n$. Lemma 8.5 states that all 7-vertex prime vertex-minors of H_n are F_1, F_2, and F_3. Lemma 8.6 proves that none of them are locally equivalent to C_7. Thus H_n has no vertex-minor isomorphic to C_7 and therefore $K_n \boxminus K_n$ has no vertex-minor isomorphic to C_7. □

9. DISCUSSIONS

9.1. **Vertex-minor ideals.** A set I of graphs is called a *vertex-minor ideal* if for all $G \in I$, all graphs isomorphic to a vertex-minor of G are also contained in I. We can interpret theorems in this paper in terms of vertex-minor ideals as follows. This formulation allows us to appreciate why these theorems are optimal.

Corollary 9.1. *Let I be a vertex-minor ideal.*

- **Theorem 3.1:** *Graphs in I have bounded number of vertices if and only if $\{\overline{K_n} : n \geqslant 3\} \nsubseteq I$.*
- **Theorem 3.1:** *Connected graphs in I have bounded number of vertices if and only if $\{K_n : n \geqslant 3\} \nsubseteq I$.*

Theorem 3.1: *Graphs in I have bounded number of edges if and only if* $\{K_n : n \geqslant 3\} \nsubseteq I$ *and* $\{\overline{K_n} \boxminus \overline{K_n} : n \geqslant 1\} \nsubseteq I$.

Theorem 7.1: *Prime graphs in I have bounded number of vertices if and only if* $\{C_n : n \geqslant 3\} \nsubseteq I$ *and* $\{K_n \boxminus K_n : n \geqslant 3\} \nsubseteq I$.

9.2. **Rough structure.** We can also regard Theorem 7.1 as a rough structure theorem on graphs having no vertex-minor isomorphic to C_n or $K_n \boxminus K_n$ as follows. The 1-*join* of two graphs G_1, G_2 with two specified vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ is the graph obtained by making the disjoint union of $G_1 \backslash v_1$ and $G_2 \backslash v_2$ and adding edges to join neighbors of v_1 in G_1 with neighbors of v_2 in G_2.

Corollary 9.2. *For each n, there exists N such that every graph having no vertex-minor isomorphic to C_n or $K_n \boxminus K_n$ can be built from graphs on at most N vertices by repeatedly taking* 1*-join operation.*

Acknowledgment

This research was done while the authors were visiting University of Hamburg. The authors would like to thank Reinhard Diestel for hosting them.

References

[1] A. Bouchet. Isotropic systems. *European J. Combin.*, 8(3):231–244, 1987.

[2] A. Bouchet. Reducing prime graphs and recognizing circle graphs. *Combinatorica*, 7(3):243–254, 1987.

[3] A. Bouchet. Graphic presentations of isotropic systems. *J. Combin. Theory Ser. B*, 45(1):58–76, 1988.

[4] A. Bouchet. Connectivity of isotropic systems. In *Combinatorial Mathematics: Proceedings of the Third International Conference (New York, 1985)*, volume 555 of *Ann. New York Acad. Sci.*, pages 81–93, New York, 1989. New York Acad. Sci.

[5] A. Bouchet. Circle graph obstructions. *J. Combin. Theory Ser. B*, 60(1):107–144, 1994.

[6] C. Chun, G. Ding, B. Oporowski, and D. Vertigan. Unavoidable parallel minors of 4-connected graphs. *J. Graph Theory*, 60(4):313–326, 2009.

[7] W. H. Cunningham. Decomposition of directed graphs. *SIAM J. Algebraic Discrete Methods*, 3(2):214–228, 1982.

[8] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[9] G. Ding and P. Chen. Unavoidable doubly connected large graphs. *Discrete Math.*, 280(1-3):1–12, 2004.

[10] G. Ding, B. Oporowski, J. Oxley, and D. Vertigan. Unavoidable minors of large 3-connected binary matroids. *J. Combin. Theory Ser. B*, 66(2):334–360, 1996.

[11] C. P. Gabor, K. J. Supowit, and W. L. Hsu. Recognizing circle graphs in polynomial time. *J. Assoc. Comput. Mach.*, 36(3):435–473, 1989.

[12] J. Geelen and S. Oum. Circle graph obstructions under pivoting. *J. Graph Theory*, 61(1):1–11, 2009.
[13] J. F. Geelen. *Matchings, matroids and unimodular matrices*. PhD thesis, University of Waterloo, 1995.
[14] W. Naji. Reconnaissance des graphes de cordes. *Discrete Math.*, 54(3):329–337, 1985.
[15] B. Oporowski, J. Oxley, and R. Thomas. Typical subgraphs of 3- and 4-connected graphs. *J. Combin. Theory Ser. B*, 57(2):239–257, 1993.
[16] S. Oum. Rank-width and vertex-minors. *J. Combin. Theory Ser. B*, 95(1):79–100, 2005.
[17] S. Oum. Approximating rank-width and clique-width quickly. *ACM Trans. Algorithms*, 5(1):Art. 10, 20, 2008.
[18] S. Oum and P. Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006.
[19] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, s2-30:264–286, 1930.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu Daejeon, 305-701 South Korea
E-mail address: ojoung@kaist.ac.kr
E-mail address: sangil@kaist.edu