Deformations of Hyperbolic Coxeter Orbifolds

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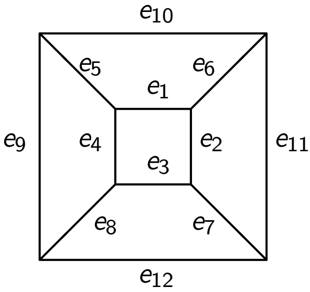
Abstract

Hyperbolic structures are examples of real projective structures if we use Klein's model. This talk was motivated by the following question: Under what conditions can one take the hyperbolic structure on a 3-dimensional hyperbolic reflection orbifold and deform it to a family of real projective structures? We will explain the numerical results in the cases of cubes and dodecahedra.

Outline

Main Results

- For reflection orbifold based on a cube with edge orders $e_1, \ldots, e_{12}$, i.e. dihedral angle $= \pi/e_i$.
- Deform a hyperbolic structure to real projective structures.



Main Results

$(e_1, e_2, \ldots, e_{11}, e_{12})$	$4f - n$	$4f -$ rank	actual
$(2, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3)$	1	1	1
$(2, 3, 2, 3, 2, 3, 3, 3, 2, 3, 2, 3)$	1	1	1
$(2, 3, 2, 3, 2, 3, 3, 3, 3, 3, 2, 2)$	1	1	1
$(2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$	0	1	0
$(2, 3, 2, 3, 3, 3, 3, 3, 2, 3, 2, 3)$	2	3	?
$(2, 3, 3, 2, 2, 3, 2, 3, 3, 3, 2, 2)$	0	1	1

- $(4f - n) = (\#$ of variables$) - (\#$ of polynomial equations$)$.
- $(4f -$ rank$) =$ (dim. of infinitesimal deformations of projective structures).
- (actual) $=$ (dim. of actual deformations of projective structures).

Coxeter Polyhedra

- X is an n-dimensional space of constant curvature.
- $\mathrm{Isom}\, X$ is the group of its motions.
- H_i^- is half-space bounded by the hyperplane H_i.
- A convex polyhedron

$$P = \bigcap_{i \in I} H_i^-$$

  is said to be a Coxeter polyhedron if for all $i, j, i \neq j$, such that the hyperplanes H_i and H_j intersect, the dihedral angle $H_i^- \cap H_j^-$ is a submultiple of π.
- The hyperplanes of $(n-1)$-dimensional faces of a convex polyhedron are said to be its walls.

Discrete Reflection Groups

- Let P be a Coxeter polyhedron, and Γ be the group generated by reflections in its walls.
- Then Γ is a discrete group of motions, and P is its fundamental polyhedron.
- Every discrete group of motions of X generated by reflections may be obtained in this way.
- The classification of Coxeter polyhedra on the sphere and Euclidean space was obtained by Coxeter in 1934.
- F. Lannér first enumerated compact Coxeter simplices in the hyperbolic space $\mathbb{H}^n$ for any n in 1950.
- They exist in $\mathbb{H}^n$ only for $n \leq 4$.

Andreev's theorem

- A nice property of a Coxeter polyhedron is that its dihedral angles are non-obtuse.
- In 1970, E.M. Andreev gave a complete description of compact 3-dimensional hyperbolic polyhedra with non-obtuse dihedral angles.
- C is an abstract 3-dimensional polyhedron.
- C^* is its dual.
- A simple closed curve γ is called a k-circuit if γ is formed of k edges of C^*.
- A circuit is called a prismatic k-circuit if all of the endpoints of the edges of C intersected by γ are distinct.
- A combinatorial polyhedron C is simple if it has no prismatic 3-circuits.

Andreev's theorem I

- Let C be an abstract polyhedron, but not a simplex. The following conditions (1) – (4) are necessary and sufficient for the existence of a compact hyperbolic polyhedron P in 3-dimensional hyperbolic space with dihedral angles not greater than $\pi/2$ such that $\alpha_{ij}(P) = \alpha_{ij}$, where F_i are the faces and α_{ij} = dihedral angle between F_i and F_j.

  1. If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then
$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} > \pi.$$
  2. If F_i, F_j, F_k generate a prismatic 3-circuit, then
$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi.$$

Andreev's theorem II

3. If F_i, F_j, F_k, F_l generate a prismatic 4-circuit, then

$$\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi.$$

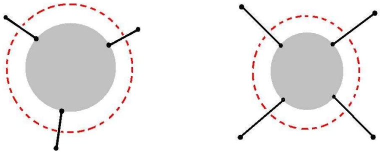


Figure: Prismatic 3-circuit & 4-circuit

Andreev's theorem III

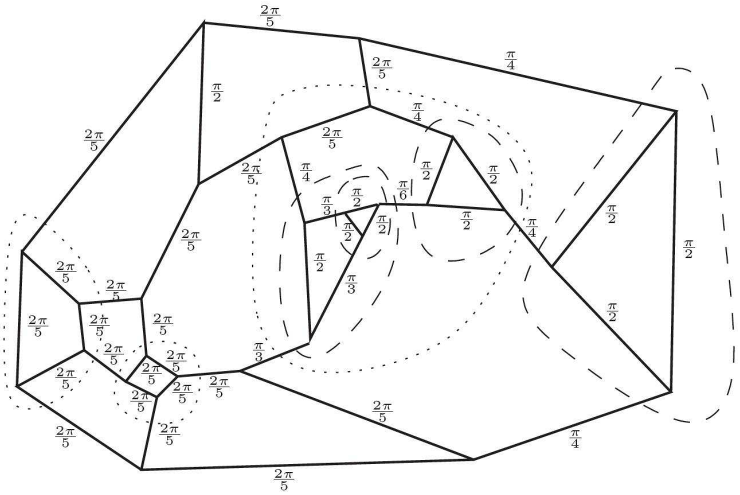
4. If F_s is a four-sided face with edges F_{is}, F_{js}, F_{ks}, F_{ls} enumerated successively, then

$$\alpha_{is} + \alpha_{ks} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi,$$

$$\alpha_{js} + \alpha_{ls} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi.$$

- Furthermore, this polyhedron is unique up to hyperbolic isometries.
- Also, it can be computed using a computer program by Roeder.
- His construction uses Newton's method and a homotopy to explicitly follow the existence proof presented by Andreev.

Roeder's program



Generalization of Reflection groups in Three Geometries

- Every simply connected space X of constant curvature can be imbedded as a hypersurface in a vector space V.

$$\begin{aligned} x_0^2 + x_1^2 + \cdots + x_n^2 = 1 & \quad \text{for} \quad \mathbb{S}^n \\ x_0 = 1 & \quad \text{for} \quad \mathbb{E}^n \\ x_0^2 - x_1^2 - \cdots - x_n^2 = 1, x_0 > 0 & \quad \text{for} \quad \mathbb{H}^n. \end{aligned}$$

- Under this imbedding, the k-dimensional planes of X are exactly the intersections of the $(k+1)$-dimensional subspace of V with X.
- In particular, every hyperplane H in X is the intersection of X and an n-dimensional subspace U of V, and
- the halfspaces in X bounded by H correspond in a natural manner to the halfspaces in V bounded by U.

Gram matrix of Hyperbolic polyhedron

- Consider for each i the unit vector e_i of the Lorentzian n-space $\mathbb{R}^{n,1}$ orthogonal to the subspace H_i and directed away from P.
- This means that the polyhedron P is the intersection in $\mathbb{R}^{n,1}$ of the convex polyhedral cone

$$K(P) = \{x \in \mathbb{R}^{n,1} \mid \langle x, e_i \rangle \leq 0,\ i = 1, \ldots, m\}$$

with X.
- The Gram matrix of the system of vector $\{e_1, \ldots, e_m\}$ is said to be the Gram matrix of the polyhedron P.
- Any indecomposable symmetric matrix of signature $(n, 1)$ with 1's along the main diagonal and non-positive entries off it is the Gram matrix for the unique convex polyhedron in the hyperbolic space $\mathbb{H}^n$ up to a motion.

Coxeter Groups

- A group with a set $\{r_1, \dots, r_m\}$ of generators is called an (abstract) Coxeter group if it has the following set of defining relations :
$$r_i^2 = 1 \text{ for all } i,$$
$$(r_i r_j)^{n_{ij}} = 1 \text{ for some } i \text{ and } j \text{ with } n_{ij} = n_{ji} \geq 2.$$
- A linear transformation R of a vector space V is a reflection if $R^2 = 1$ and -1 is a simple eigenvalue of R.
- A reflection R is completely determined by the subspace U of fixed points and an eigenvector b corresponding to the eigenvalue -1.

Linear Coxeter Groups

- If α is the linear functional on V vanishing on U and such that $\alpha(b) = 2$, then

$$Rv = v - \alpha(v)b.$$

- Let K be a convex polyhedral cone in V defined by a system of linear inequalities

$$\alpha_i \geq 0, \qquad i = 1, \ldots, m,$$

  and suppose that none of these inequalities follows from the remaining ones.
- Let $b_i (i = 1, \ldots, m)$ be elements of V satisfying $\alpha_i(b_i) = 2$.
- Let R_i be the reflection defined by above for $\alpha = \alpha_i$, $b = b_i$.

Vinberg's Main Results

- The group $\Gamma \subset GL(V)$ generated by the R_i will be called a discrete linear group generated by reflections, or simply a linear Coxeter group if

$$\gamma K^0 \cap K^0 = \varnothing \text{ for every } \gamma \in \Gamma \backslash \{1\}.$$

- K will be called a fundamental chamber of Γ.
- The Cartan matrix of Γ is the $m \times m$ matrix $A = (a_{ij})$, $a_{ij} = \alpha_i(b_j)$.
- Γ is a linear Coxeter group if and only if the Cartan matrix of Γ satisfies the following conditions.

(C1) $a_{ij} \leq 0$ for $i \neq j$ and if $a_{ij} = 0$ then $a_{ji} = 0$.
(C2) $a_{ii} = 2$; $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2 \frac{\pi}{n_{ij}}$, n_{ij} an integer.

Orthogonality of Linear Coxeter Groups

- Let X simply connected space of constant curvature.
- Every discrete group generated by reflections in X operates on V as a linear Coxeter group.
- A linear Coxeter group Γ is orthogonal if there exists a Γ-invariant scalar product in the subspace of V spanned by the b_i such that $(b_i, b_i) > 0$ for all i.
- All those linear Coxeter groups obtained from groups generated by reflections in spaces of constant curvature are orthogonal.

Orthogonality of linear Coxeter groups

- Matrices A and B will be called equivalent if $A = DBD^{-1}$ for a diagonal matrix D having positive diagonal elements.
- For distinct values of $i_1, \cdots, i_k$ the expressions

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1},$$

  will be called cyclic products of $A = (a_{ij})$.
- Matrices A and B satisfying $(C1)$ are equivalent if and only if their cyclic products are identical.
- A linear Coxeter group is orthogonal if and only if its Cartan matrix A is equivalent to the corresponding symmetric matrix.

The properties of the linear Coxeter groups

- Let Γ be a discrete linear group generated by reflections $R_1, \cdots, R_m$ in the faces of a convex polyhedral cone K.
- For any $x \in K$ let Γ_x denote the subgroup of Γ generated by reflections in those faces of K which contain x.
- Define $K^f = \{x \in K \mid \Gamma_x \text{ is finite}\}$.
- Then the following assertions are true.

The properties of the linear Coxeter groups

1. $\cup_{\gamma \in \Gamma} \gamma K$ is a convex cone.
2. Γ operates discretely in the interior C of this cone.
3. $C \cap K = K^f$
4. The canonical map from K^f to C/Γ is a homeomorphism.
5. For every $x \in K$, Γ_x is the stabilizer of x in Γ.
6. For every pair of adjacent faces K_i, K_j of K, let n_{ij} denote the order of $R_i R_j$ (n_{ij} may be infinite). Then

$$R_i^2 = 1, \quad (R_i R_j)^{n_{ij}} = 1.$$

is a system of defining relations for Γ.

Coxeter Orbifold Structures

- Let P be a fixed 3-dimensional convex polyhedron.
- Let us assign orders at each edge.
- Let e be the number of edges and e_2 be the numbers of edges of order-two. Let f be the number of faces.
- We remove any vertex of P which has more than three edges incident or with orders of the edges incident not of form

$$(2, 2, n),\ n \geq 2,\ (2, 3, 3),\ (2, 3, 4),\ (2, 3, 5),$$

  i.e., orders of spherical triangular groups.
- This makes P into an open 3-dimensional orbifold.
- Let $\hat{P}$ denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from P with vertices removed as above.

Normal-type Coxeter Orbifold

- We say that $\hat{P}$ has a Coxeter orbifold structure.
- We will not study
  1. Cone-type Coxeter orbifolds whose polyhedron has a face F and a vertex v where all other sides are adjacent triangles to F and contains v and all edge orders of F are 2.
  2. Product-type Coxeter orbifolds whose polyhedron is topologically a polygon times an interval and edge orders of top and the bottom faces are all 2.

     These are essentially two-dimensional orbifolds which can be better studied by more elementary methods.
  3. Coxeter orbifolds with finite fundamental groups.
- If $\hat{P}$ is none of the above type, then $\hat{P}$ is said to be a normal-type Coxeter orbifold.

Deformation spaces of Projective Structures

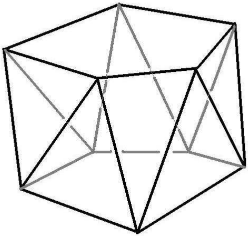
- An isotopy of an orbifold M is an orbifold-diffeomorphism $f : M \to M$ such that there exists an orbifold map $H : M \times I \to M$ which restricts to an identity for $t = 0$ and restricts to f for $t = 1$.
- The deformation space $\mathfrak{D}(\hat{P})$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ up to orbifold isotopy.
- A point p of $\mathfrak{D}(\hat{P})$ always determines a fundamental polyhedron P up to projective automorphisms.

Restricted Deformation Space

- We wish to understand the space where the fundamental polyhedron is always projectively equivalent to a fixed P.
- We call this the restricted deformation space of $\hat{P}$ and denoted it by $\mathfrak{D}_P(\hat{P})$.
- We say that P is orderable if we can order the faces of P so that each face meets faces of higher index in less than or equal to 3 edges.
- Examples are
  0. Convex polyhedron with all faces triangular.

Examples of orderable polyhedra

1. Drum-shaped convex polyhedron which has top and bottom faces of same polygonal type and each vertex of the bottom face is connected to two vertices in the top face and vice versa.



2. Convex polyhedron where each pair of the interiors of nontriangular faces are separated by a union of triangles.

- A cube and dodecahedron would not satisfy the conditions.

Orderable Orbifolds

- Let $\hat{P}$ be the orbifold obtained from P by silvering faces and removing vertices as above.
- We also say that the orbifold $\hat{P}$ is orderable if the faces of P can be ordered so that each face has no more than three edges which are either of order 2 or included in a face of higher index.
- $k(P)$ is the dimension of the group of projective automorphism of a convex polyhedron P.

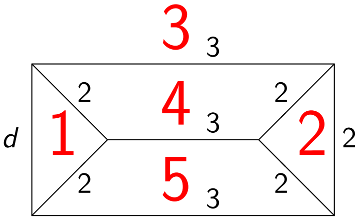
$$k(P) = \begin{cases} 3 & \text{if } P \text{ is a tetrahedron,} \\ 1 & \text{if } P \text{ is a pyramid with base} \\ & \text{a convex polygon which is not a triangle,} \\ 0 & \text{otherwise} \end{cases}$$

Choi's Main result

- Let P be a convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure.
- Let $k(P)$ be the dimension of the group of projective automorphisms of P.
- Suppose that $\hat{P}$ is orderable.
- Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.

Benoist's Example

- Let $d = 3$, 4, or 5.



- This triangular prism has no spherical or Euclidean or hyperbolic structure.
- This is orderable. (See the numbers of red color.)
- $3f - e - e_2 - k(P) = 15 - 9 - 5 - 0 = 1.$
- The dimension of $\mathfrak{D}_P(\hat{P}) = 1$.

Construction of hyperbolic Coxeter polyhedra

- V is a 4-dimensional vector space over $\mathbb{R}$ with coordinates $x_1, \ldots, x_4$.
- P is a Coxeter polyhedron in Klein's model of 3-dimensional hyperbolic space.
- In other words, P is given by a system of linear inequalities

$$\alpha_i \geq 0, \quad i = 1, \ldots, f, \text{ and } x_1 = 1$$

  where $\alpha_i : V \to \mathbb{R}$ are linear functionals on V.
- For simplicity, we assume that P is neither tetrahedron nor pyramid.

Construction of hyperbolic Coxeter polyhedra

- Lorentzian inner product is denoted by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 .$$

- The problem of constructing a polyhedron P can be expressed as the problem of finding a solution to the following equations:

$$\langle \alpha_i, \alpha_i \rangle = 1, \text{ for all } i$$

$$\langle \alpha_i, \alpha_j \rangle = -\cos(\pi / n_{ij}) \text{ if faces } F_i \text{ and } F_j \text{ are adjacent in } P.$$

- In general, it is difficult to find an exact algebraic solution.
- Roeder's Matlab program can be used to obtain a numerical solution.

Vinberg's equations

- After finding it, α_i's will be *fixed*.
- $b_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4})$ for all $1 \leq i \leq f$ are the reflection points, i.e. eigenvector corresponding to the eigenvalue -1.
- In fact, b_i are *variables*.
- R_i are the reflections defined by

$$R_i(v) = v - \alpha_i(v) b_i,$$

  where $\alpha_i(b_i) = 2$.
- The group $\Gamma \subset GL(V)$ is generated by R_i.
- The matrix $A = (a_{ij}) = (\alpha_i(b_j))$ is the $f \times f$ Cartan matrix of Γ.

Variety

- Vinberg's result forces us to solve the system $\{\Phi_k = 0\}_{k=1}^n$ of polynomial equations, where $n = f + e + e_2$ and

$$\{\Phi_k\} = \{a_{ii} - 2,\ \underbrace{a_{ij}a_{ji} - 4\cos^2(\pi/n_{ij})}_{\text{if } n_{ij}\neq 2},\ \underbrace{a_{ij},\ a_{ji}}_{\text{if } n_{ij}=2}\}.$$

- The map $\Phi : \mathbb{R}^{4f} \to \mathbb{R}^n$ is given by

$$(b_1, \ldots, b_f) \mapsto (\Phi_1, \ldots, \Phi_n).$$

- The variety $\Phi^{-1}(0)$ is what we want to know.
- Abuse of notation: $V = V^* = V^{**}$.
- $b_i = 2J\alpha_i$ gives the solution $S = \{S_i\} = \{2J\alpha_i\}$ which is related to a hyperbolic structure, where $J = \text{diag}(-1, 1, 1, 1)$, for $a_{ij} = \alpha_i(2J\alpha_j) = 2\langle\alpha_i, \alpha_j\rangle = 2\langle\alpha_j, \alpha_i\rangle = a_{ji}$.

Summary of the Computational procedure

1. Construct hyperbolic Coxeter polyhedron by Roeder's Matlab program.
2. Compute Zariski tangent space at hyperbolic point.
3. If it is not sufficient to check the tangent space to calculate the local dimension of the restricted deformation space, then we try to find the Gröbner basis of the ideal $\{\Phi_k = 0\}_{k=1}^{n}$.

Jacobian matrix

- $D = (d_{lm})$ is the $n \times 4f$ jacobian matrix given by

$$d_{lm} = \left.\frac{\partial \Phi_k}{\partial b_{ij}}\right|_{\{b_i\}=S}.$$

- If $\Phi_k = a_{ii} - 2 = \alpha_i(b_i) - 2$
$= \alpha_{i1}b_{i1} + \alpha_{i2}b_{i2} + \alpha_{i3}b_{i3} + \alpha_{i4}b_{i4} - 2$, then

$$\begin{aligned}\left.\frac{\partial \Phi_k}{\partial b_{ij}}\right|_{\{b_i\}=S} &= (0,\ldots,0,\alpha_{i1},\alpha_{i2},\alpha_{i3},\alpha_{i4},0,\ldots,0)\\ &= (0,\ldots,0,\underbrace{\alpha_i}_{i\text{ block}},0,\ldots,0).\end{aligned}$$

Jacobian matrix

- If $\Phi_k = a_{ij}a_{ji} - 4\cos^2(\pi/n_{ij})$
  $= (\alpha_{i1}b_{j1} + \cdots + \alpha_{i4}b_{j4})(\alpha_{j1}b_{i1} + \cdots + \alpha_{j4}b_{i4}) - 4\cos^2(\pi/n_{ij})$,
  then

$$\begin{aligned} \left.\frac{\partial \Phi_k}{\partial b_{ij}}\right|_{\{b_i\}=S} &= (0,\ldots,0,\underbrace{a_{ij}\alpha_j}_{i\text{ block}},0,\ldots,0,\underbrace{a_{ji}\alpha_i}_{j\text{ block}},0,\ldots,0) \\ &= a_{ij}(0,\ldots,0,\underbrace{\alpha_j}_{i\text{ block}},0,\ldots,0,\underbrace{\alpha_i}_{j\text{ block}},0,\ldots,0), \end{aligned}$$

- since $a_{ij} = a_{ji}$ at a hyperbolic point.

Jacobian matrix

- If $\Phi_k = a_{ij} = \alpha_{i1} b_{j1} + \cdots + \alpha_{i4} b_{j4}$, then

$$\left.\frac{\partial \Phi_k}{\partial b_{ij}}\right|_{\{b_i\}=S} = (0, \ldots, 0, \underbrace{0}_{i \text{ block}}, 0, \ldots, 0, \underbrace{\alpha_i}_{j \text{ block}}, 0, \ldots, 0).$$

- If $\Phi_k = a_{ji} = \alpha_{j1} b_{i1} + \cdots + \alpha_{j4} b_{i4}$, then

$$\left.\frac{\partial \Phi_k}{\partial b_{ij}}\right|_{\{b_i\}=S} = (0, \ldots, 0, \underbrace{\alpha_j}_{i \text{ block}}, 0, \ldots, 0, \underbrace{0}_{j \text{ block}}, 0, \ldots, 0).$$

- In other words, $n_{ij} = 2$ means a row splits in two.

Zariski Tangent space at Hyperbolic point

- The Zariski tangent space is the kernel of Jacobian matrix D.
- Let $\mathfrak{D}_P(\hat{P})$ denote the restricted deformation space of projective structures on a Coxeter orbifold $\hat{P}$.
- If $4f - n > 0$ and D has full rank, then *the neighborhood of S in $\mathfrak{D}_P(\hat{P})$ has locally $(4f - n)$-dimensional differentiable structure.*
- So the hyperbolic structure on a Coxeter orbifold deforms to a nontrivial real projective structure.
- If $4f - n \leq 0$ and D has full rank, then the hyperbolic structure is rigid.
- Note that $4f - n = 4f - (f + e + e_2) = 3f - e - e_2$.
- These observations lead to Choi's result in the *orderable* case.

Ideal hyperbolic polyhedron

- Assume P is a convex ideal hyperbolic polyhedron, i.e. one with all vertices on the sphere at infinity, of which every edge has order 3.
- If P is simple, then there exists a hyperbolic polyhedron which satisfies the above conditions.
- We return to the problem of constructing a hyperbolic polyhedron P.
- It is same as solving the system $\{\Psi_k = 0\}_{k=1}^n$ of polynomial equations, where $n = f + e$ and

$$\{\Psi_k\} = \{\langle \alpha_i, \alpha_i \rangle - 1, \langle \alpha_i, \alpha_j \rangle + \cos(\pi / n_{ij})\}.$$

- This has a unique solution S up to hyperbolic isometries.

Ideal hyperbolic polyhedron

- $\hat{D} = (\hat{d}_{lm})$ is the $n \times 4f$ jacobian matrix given by

$$\hat{d}_{lm} = \left.\frac{\partial \Psi_k}{\partial \alpha_{ij}}\right|_{\{\alpha_i\}=S}.$$

- If $\Psi_k = \langle \alpha_i, \alpha_i \rangle - 1 = -\alpha_{i1}^2 + \alpha_{i2}^2 + \alpha_{i3}^2 + \alpha_{i4}^2 - 1$, then

$$\begin{aligned} \left.\frac{\partial \Psi_k}{\partial \alpha_{ij}}\right|_{\{\alpha_i\}=S} &= (0, \ldots, 0, -2\alpha_{i1}, 2\alpha_{i2}, 2\alpha_{i3}, 2\alpha_{i4}, 0, \ldots, 0) \\ &= (0, \ldots, 0, \underbrace{2J\alpha_i}_{i \text{ block}}, 0, \ldots, 0). \end{aligned}$$

Ideal hyperbolic polyhedron

- If $\Psi_k = \langle \alpha_i, \alpha_j \rangle + \cos(\pi/n_{ij})$
$= -\alpha_{i1}\alpha_{j1} + \alpha_{i2}\alpha_{j2} + \alpha_{i3}\alpha_{j3} + \alpha_{i4}\alpha_{j4} + \cos(\pi/n_{ij})$, then

$$\left.\frac{\partial \Psi_k}{\partial \alpha_{ij}}\right|_{\{\alpha_i\}=S} = (0, \dots, 0, \underbrace{J\alpha_j}_{i \text{ block}}, 0, \dots, 0, \underbrace{J\alpha_i}_{j \text{ block}}, 0, \dots, 0).$$

- Hence the rank of D and $\hat{D}$ are equal.
- Then we have: $3v = 2e, v - e + f = 2$. So $3f - e = 6$, i.e. the dimension of the group of hyperbolic isometries.
- Rigidity of hyperbolic orbifold $\Rightarrow$ $\hat{D}$ has full rank.
- *The neighborhood of S in $\mathfrak{D}_P(\hat{P})$ has locally 6-dimensional differentiable structure.*

Some Properties

- Ideal hyperbolic polyhedron must pass through the plane which contains reflection points of three faces intersecting at a vertex during deformation.

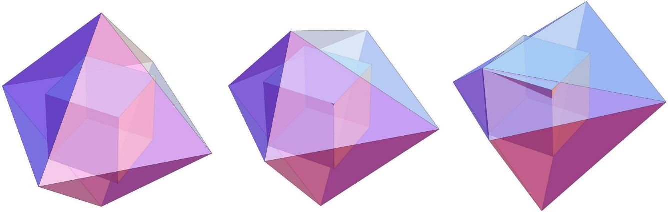
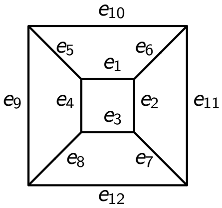


Figure: Deformations of ideal cubes

Compact hyperbolic cubes

- P is a compact hyperbolic cube all of whose dihedral angles are $\pi/2$ or $\pi/3$.
- Assume that each face has less than three edges which are of order 2.
- If a face F_i has more than two edges $\{F_{ij}, F_{ik}, F_{il}\}$ which are of order 2, then b_i is not a variable anymore.
- The system $\{\alpha_i(b_i) = 2, \alpha_j(b_i) = 0, \alpha_k(b_i) = 0, \alpha_l(b_i) = 0\}$ of linear equations fixes b_i.
- The total number of such cubes is 6 (up to symmetries).

Table for cubes



$(e_1, e_2, \ldots, e_{11}, e_{12})$	$4f - n$	$4f$ − rank	actual
(2, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3)	1	1	1
(2, 3, 2, 3, 2, 3, 3, 3, 2, 3, 2, 3)	1	1	1
(2, 3, 2, 3, 2, 3, 3, 3, 3, 3, 2, 2)	1	1	1
(2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3)	0	1	0
(2, 3, 2, 3, 3, 3, 3, 3, 2, 3, 2, 3)	2	3	?
(2, 3, 3, 2, 2, 3, 2, 3, 3, 3, 2, 2)	0	1	1

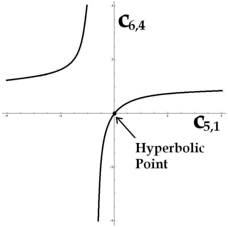
- 'actual' means 'dimension of actual deformations'.
- The information on the three cubes in the upper half of the table is determined by computing the jacobian matrices.

Actual Deformations

- It is not sufficient only to check jacobian matrix D to calculate the local dimension of $\mathfrak{D}_P(\hat{P})$.
- Introduce new coordinates on $\mathbb{R}^{4f}$ by letting $c_i = b_i - S_i$.
- Relative to this coordinate system, the hyperbolic point S is the origin.
- We compute a Gröbner basis G of the ideal $I = \langle \Phi_k \rangle$ with respect to the lex order with $c_{6,4} > c_{6,3} > c_{6,2} > c_{6,1} > c_{5,4} > \cdots$.
- Gröbner basis of the fourth cube is $\{c_{6,4}, c_{6,3}, c_{6,2}^2, c_{6,1}, c_{5,4}, \ldots\}$.
- Gröbner basis of the radical ideal $\sqrt{I}$ is $\{c_{6,4}, c_{6,3}, c_{6,2}, c_{6,1}, c_{5,4}, \ldots, c_{1,2}, c_{1,1}\}$.
- So, there are no non-trivial deformations.

Gröbner bases

- However, Gröbner basis of the last cube is $\{c_{6,3} + c_{6,4},\ c_{6,2} + c_{6,4}, \sqrt{7}c_{6,1} + 5c_{6,4},\ c_{5,4},\ c_{5,3},\ c_{5,2},$ $\boldsymbol{\sqrt{7}c_{5,1} - 2c_{6,4} - \sqrt{7}c_{5,1}c_{6,4}},\ c_{4,4},\ c_{4,3},\ c_{4,2},\ c_{4,1} - c_{5,1},$ $c_{3,4} - c_{6,4},\ c_{3,3} + c_{6,4},\ c_{3,2} + c_{6,4},\ \sqrt{7}c_{3,1} + 5c_{6,4},\ c_{2,4} - c_{6,4},$ $c_{2,3} + c_{6,4},\ c_{2,2} + c_{6,4},\ \sqrt{7}c_{2,1} + 5c_{6,4},\ c_{1,4},\ c_{1,3},\ c_{1,2},\ c_{1,1} - c_{5,1}\}$
- So, the actual dimension of deformations is 1.
- Unfortunately for the fifth cube, we cannot currently compute a Gröbner basis, since the coefficients of Φ_k are quite complicated.



Dodecahedron

- P is a compact hyperbolic dodecahedron all of whose dihedral angles are $\pi/2$ or $\pi/3$.
- Assume that each face has less than three edges which are of order 2.
- The total number of such dodecahedra is 13 (up to symmetries).

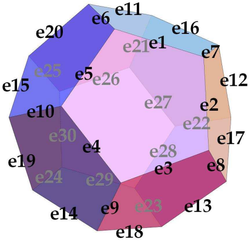


Table for Dodecahedron

- Here is a table for these.

$(e_1, e_2, \ldots, e_{29}, e_{30})$	O	Z	A
(2, 3, 2, 3, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 3, 3, 3, 3, 3, 2, 3, 2, 3, 2, 3, 3, 2, 3, 2)	-6	0	0
(2, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 2, 2, 2, 3, 3, 3, 3, 3, 3, 2, 3, 3, 2, 2, 3, 3, 2, 3, 3)	-5	0	0
(2, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 2, 2, 2, 3, 3, 3, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 2, 3)	-5	0	0
(2, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 2, 2, 2, 3, 3, 3, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 2)	-5	0	0
(2, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 2, 3, 2, 3, 3, 2, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 2)	-6	0	0
(2, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 2, 3, 3, 3, 3, 2, 2, 3, 3, 2, 3, 3, 2, 2, 3, 3, 2, 3, 3)	-5	0	0
(2, 3, 2, 3, 3, 3, 2, 2, 3, 2, 3, 3, 3, 2, 3, 3, 3, 3, 3, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 2)	-5	0	0
(2, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 2, 2, 2, 3, 3, 3, 3, 3, 2, 2, 3, 3, 2, 3, 3, 3, 2, 3, 2)	-6	0	0
(2, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 2, 2, 2, 3, 3, 3, 3, 3, 2, 2, 3, 3, 3, 3, 3, 3, 2, 3, 2)	-5	0	0
(2, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 2, 2, 3, 3, 3, 3, 3, 2, 2, 2, 3, 3, 3, 3, 3, 3, 2, 2)	-5	0	0
(2, 3, 2, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 3, 2, 3, 2, 3)	-4	0	0
(2, 3, 2, 3, 3, 3, 3, 3, 3, 2, 2, 2, 3, 3, 2, 3, 2, 2, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3)	-6	0	0
(2, 3, 2, 3, 3, 3, 3, 3, 3, 2, 3, 3, 2, 2, 3, 2, 3, 3, 3, 2, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3)	-4	1	?

where 'O'$= 4f - n$, 'Z'$=4f -$ rank, and 'A'=dimension of actual deformations.

- The information on all dodecahedra in the table is determined by computing the jacobian matrices.

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