## Announcements:

The 8th KAIST Geometric Topology Fair January 11-13, 2010
KAIST, Daejeon
http://mathsci.kaist.ac.kr/~manifold/8thgtfair.html
Hyperbolic geometry: algorithmic, number theoretic and numerical
aspects (A graduate student workshop)
March 15-19, 2010
KIAS, Seoul
Main Lecturers: Craig Hodgson, Walter Neumann, Alan Reid http://mathsci.kaist.ac.kr/~schoi/hyperbolic.html

# Projective Deformations <br> of 3-dimensional Hyperbolic Coxeter Orbifolds 

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## Abstract

By Andreev's theorem, many 3-dimensional reflection orbifolds admit finite volume hyperbolic structures.

Mostow-Prasad rigidity $\Longrightarrow$ Such a hyperbolic structure is unique. By using Klein's model, hyperbolic structures on orbifolds provide examples of induced real projective structures.

The induced real projective structure on some 3-orbifolds deforms into a family of real projective structures that are not hyperbolic.

We find such a class of complete hyperbolic reflection orbifolds.
We also explain numerical and exact results on projective deformations of some compact hyperbolic cubes and dodecahedra.

We would like to thank M. Kapovich for his helpful comments.

## Coxeter orbifold structure

- We focus on 3-dimensional reflection orbifolds whose underlying space is homeomorphic to a convex polyhedron, and whose singular locus is its boundary.
- The fundamental group of such an orbifold is a Coxeter group;

$$
<r_{1}, \ldots, r_{m} \mid\left(r_{i} r_{j}\right)^{n_{i j}}=1>
$$

- Let $P$ be a fixed 3-dimensional convex polyhedron, and assign an order $n_{e} \geq 2$ to each edge $e$ of $P$.
- If any vertex of $P$ has more than three edges incident, or has orders of the incident edges not of the form

$$
(2,2, k) \text { with } k \geq 2, \quad(2,3,3), \quad(2,3,4), \quad(2,3,5)
$$

then we remove the vertex.

- $\hat{P}$ is the differentiable orbifold obtained from $P$ with faces silvered, edge orders $n_{e}$, and with vertices removed as above.
- We say that $\hat{P}$ has a Coxeter orbifold structure.


## Deformation spaces

- The orbifold $\hat{P}$ is a normal-type Coxeter orbifold if it is not

1. a cone-type Coxeter orbifold,
2. a product-type Coxeter orbifold,
3. a Coxeter orbifolds with finite fundamental group.

- We restrict ourselves to normal-type orbifolds.
- The deformation space $\mathfrak{D}(\hat{P})$ of real projective structures on the orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ up to orbifold isotopies.
- A point $p$ of $\mathfrak{D}(\hat{P})$ gives a fundamental polyhedron $P$ in $\mathbb{R} P^{3}$, well-defined up to projective automorphisms.
- We concentrate on the space where the point $p$ gives a fixed fundamental polyhedron $P$, which is called the restricted deformation space $\mathfrak{D}_{P}(\hat{P})$ of $\hat{P}$.


## Orderability results

- A Coxeter orbifold $\hat{P}$ is said to be orderable if the faces of $P$ can be ordered so that each face contains less than four edges which are edges of order 2 or edges in a face of higher index.
- Let $k(P)$ be the dimension of the group of projective automorphisms of $P$.

$$
k(P)= \begin{cases}3 & \text { if } P \text { is a tetrahedron, } \\ 1 & \text { if } P \text { is a pyramid with base } \\ & \text { a convex polygon which is not a triangle, } \\ 0 & \text { otherwise }\end{cases}
$$

- (Choi 2006) The restricted deformation space of projective structures on the orderable orbifold $\hat{P}$ is a smooth manifold of dimension $3 f-e-e_{2}-k(P)$ if it is not empty.
- Cubes and dodecahedra do not carry an orderable Coxeter orbifold structure.


## Restrcted deformation spaces

- Sending a development pair $(D, h)$ to the conjugacy class of the holonomy representation $h$ induces a local homeomorphism

$$
\text { hol }: \mathfrak{D}(\hat{P}) \rightarrow \operatorname{Hom}\left(\pi_{1}(\hat{P}), G\right) / G, \text { with } G=S L_{ \pm}(4, \mathbb{R})
$$

- To study restricted deformation spaces, we define

$$
\operatorname{Hom}_{P}\left(\pi_{1}(\hat{P}), G\right) \subset \operatorname{Hom}\left(\pi_{1}(\hat{P}), G\right)
$$

- $G_{P}$ is the subgroup of $G$ that preserves $P$ and each of its faces.
- Assume that $k(P)=0$. Then $G_{P}$ is trivial.
- The map $\iota_{P}: \operatorname{Hom}_{P}\left(\pi_{1}(\hat{P}), G\right) \rightarrow \operatorname{Hom}\left(\pi_{1}(\hat{P}), G\right) / G$ is an imbedding.
- The restricted map holp from hol induces a local homeomorphism near $p$

$$
\text { hol }_{P}: \mathfrak{D}_{P}(\hat{P}) \rightarrow \operatorname{Hom}_{P}\left(\pi_{1}(\hat{P}), G\right)
$$

## Local deformation spaces

- Consider the representation $h$, we say that $h$ is genuine if $h\left(r_{i}\right) h\left(r_{i}\right)$ is conjugate to a rotation of angle $2 \pi / n_{i j}$.
- The map hol $P_{P}: \mathfrak{D}_{P}(\hat{P}) \rightarrow \operatorname{Hom}_{P}^{\mathfrak{g}}\left(\pi_{1}(\hat{P}), G\right)$ is a homeomorphism.
- A point $t$ in $\mathfrak{D}_{P}(\hat{P})$ is hyperbolic if it is given by a hyperbolic structure on $\hat{P}$.
- Suppose that $t$ is the corresponding hyperbolic point of $\mathfrak{D}_{P}(\hat{P})$.
- We call a neighborhood of $t$ in $\mathfrak{D}_{P}(\hat{P})$ the local deformation space of $P$.
- We say that $P$ is deformable if the dimension of its local deformation space is positive.
- Conversely, we say that $P$ is projectively rigid, or simply rigid if the dimension of its local deformation space is 0 .


## Reflections

- A reflection $R$ on $V=\mathbb{R}^{4}$ is an element of order 2 of $G$ which is the identity on a hyperplane $U$.
- All reflections are of the form

$$
R=I d-\alpha \otimes b
$$

for some $\alpha \in V^{*}$ and $b \in V$ with $\alpha(b)=2$.

- The kernel of $\alpha$ is the subspace $U$ of fixed points of $R$.
- $b$ is the reflection vector.
- Consider $\mathbb{S}^{3}$ as the set of rays in $V$ from the origin.
- $P$ is a $n$-dimensional convex polytope in $\mathbb{S}^{3}$.
- A suitable choice of signs allows us to suppose that $P$ is defined by the inequalities

$$
\alpha_{i} \leq 0 \quad i=1, \ldots, f
$$

## Vinberg's results

- The group $\Gamma \subset G$ generated by all these reflections $R_{i}$ is called a linear Coxeter group if

$$
\gamma P^{\circ} \cap P^{\circ}=\varnothing \text { for every } \gamma \in \Gamma \backslash\{1\}
$$

- The matrix $A=\left(a_{i j}\right)=\left(\alpha_{i}\left(b_{j}\right)\right)$ is the Cartan matrix of $\Gamma$.
- The following conditions are necessary and sufficient for $\Gamma$ to be a linear Coxeter group:

$$
\begin{aligned}
& \text { 1. } a_{i j} \leq 0 \text { for } i \neq j \text {, and } a_{i j}=0 \Leftrightarrow a_{j i}=0 . \\
& \text { 2. } a_{i i}=2 ; a_{i j} a_{j i} \geq 4 \text { or } a_{i j} a_{j i}=4 \cos ^{2} \frac{\pi}{n_{i j}}, n_{i j} \text { an integer. }
\end{aligned}
$$

- For any $x \in P$, let $\Gamma_{x}$ denote the subgroup of $\Gamma$ generated by reflections in those faces of $P$ which contain $x$.
- Define $P^{f}=\left\{x \in P \mid \Gamma_{x}\right.$ is finite $\}$.
- $C=\cup_{\gamma \in \Gamma} \gamma P$ is convex.
- $\Gamma$ is a discrete subgroup of $G$ preserving $C^{\circ}$.
- $C^{\circ} \cap P=P^{f}$, and is homeomorphic to $C^{\circ} / \Gamma$.


## The space of representations

- $P$ is a fixed convex polyhedron in $\mathbb{S}^{3}$ with $k(P)=0$.
- $P$ is given by a system of linear inequalities

$$
\alpha_{i} \leq 0, \quad i=1, \ldots, f
$$

where $\alpha_{i} \in V^{*}$ and $f$ is the number of faces of $P$.

- Suppose $b_{i}=\left(b_{i 1}, b_{i 2}, b_{i 3}, b_{i 4}\right)$ for $1 \leq i \leq f$ are reflection vectors with $\alpha_{i}\left(b_{i}\right)=2$.
- Let $R_{i}$ be the reflections defined by

$$
R_{i}=I d-\alpha_{i} \otimes b_{i} \text { for } i=1, \ldots, f
$$

- $\Gamma \subset G$ is the group generated by the $R_{i}$.
- The matrix $A=\left(a_{i j}\right)$ is the $f \times f$ Cartan matrix of $\Gamma$.
- We consider the restricted deformation space of the corresponding Coxeter orbifold $\hat{P}$.
- The $\alpha_{i}$ 's will be fixed, and $b_{i}$ 's are variables.


## The space of representations

- Vinberg's result $\Longrightarrow$ Vinberg equations:

1. For each $i=1, \ldots, f, a_{i i}=\alpha_{i}\left(b_{i}\right)=2$,
2. If $F_{i}$ and $F_{j}$ are adjacent in $P$ and $n_{i j}>2$,

$$
a_{i j} a_{j i}=\alpha_{i}\left(b_{j}\right) \alpha_{j}\left(b_{i}\right)=4 \cos ^{2}\left(\pi / n_{i j}\right)
$$

3. If $F_{i}$ and $F_{j}$ are adjacent in $P$ and $n_{i j}=2$,

$$
a_{i j}=\alpha_{i}\left(b_{j}\right)=0 \text { and } a_{j i}=\alpha_{j}\left(b_{i}\right)=0
$$

- $\Phi_{P}: \mathbb{R}^{4 f} \rightarrow \mathbb{R}^{N}$ is the map given by

$$
\left(b_{1}, \ldots, b_{f}\right) \mapsto\left(\Phi_{1}, \ldots, \Phi_{N}\right)
$$

where $\left\{\Phi_{k}\right\}_{k=1}^{N}$ is the set of polynomials $a_{i i}-2$, $a_{i j} a_{j i}-4 \cos ^{2}\left(\pi / n_{i j}\right)$, or $a_{i j}, a_{j i}$ as in the above conditions.

- There is a homeomorphism

$$
\mathcal{H}: \Phi_{P}^{-1}(0) \rightarrow \operatorname{Hom}_{P}^{\mathfrak{g}}\left(\pi_{1}(\hat{P}), G\right) \cong \mathfrak{D}_{P}(\hat{P})
$$

where $\mathcal{H}$ is given by sending the coordinates to the appropriate reflections.

## The hyperbolic point

- $P$ is a Coxeter polyhedron in Klein's model of $\mathbb{H}^{3}$.
- The Lorentzian inner product on $V$ is defined by

$$
\langle x, y\rangle=-x_{1} y_{1}+x_{2} y_{2}+x_{3} y_{3}+x_{4} y_{4} .
$$

- $\nu_{i} \in V$ is the outward unit normal to the $i$ th face of $P$.
- The problem of constructing a hyperbolic polyhedron $P$ with prescribed dihedral angles $\pi / n_{i j}$ :

$$
\begin{aligned}
& \left\langle\nu_{i}, \nu_{i}\right\rangle=1 \text { for all } i=1, \ldots, f \\
& \left\langle\nu_{i}, \nu_{j}\right\rangle=-\cos \left(\pi / n_{i j}\right) \text { if faces } F_{i} \text { and } F_{j} \text { are adjacent. }
\end{aligned}
$$

- We call these equations the hyperbolic equations.
- $\nu_{i} \longleftrightarrow \alpha_{i} \in V^{*}$ given by $\alpha_{i}(x)=\left\langle\nu_{i}, x\right\rangle$.
- $b_{i}=2 \nu_{i} \longleftrightarrow$ a hyperbolic point $t=\left\{2 \nu_{i}\right\}$ in $\mathfrak{D}_{P}(\hat{P})$.
- In the space $\Phi_{P}^{-1}(0)$ of representations, there is a single point corresponding to the hyperbolic structure on $\hat{P}$.


## The Zariski tangent space

- Each $\alpha_{i} \in V^{*}$ is fixed, and the equations have the form

1. $\Phi_{i i}=\alpha_{i}\left(b_{i}\right)-2=0$
2. $\Phi_{i j}=\alpha_{i}\left(b_{j}\right) \alpha_{j}\left(b_{i}\right)-c_{i j}=0$ where $c_{i j}$ is a constant if $n_{i j} \neq 2$,
3. $\Phi_{i j}^{1}=\alpha_{i}\left(b_{j}\right)=0$ and $\Phi_{i j}^{2}=\alpha_{j}\left(b_{i}\right)=0$ if $n_{i j}=2$.

- The rows of the $N \times 4 f$ Jacobian matrix $D=[D \Phi]$ :

$$
\begin{aligned}
{\left[D \Phi_{i i}\right] } & =\left(0, \ldots, 0, \alpha_{i 1}, \alpha_{i 2}, \alpha_{i 3}, \alpha_{i 4}, 0, \ldots, 0\right) \\
& =(0, \ldots, 0, \underbrace{\alpha_{i}}_{i \text { th block }}, 0, \ldots, 0), \\
{\left[D \Phi_{i j}\right] } & =(0, \ldots, 0, \underbrace{a_{i j} \alpha_{j}}_{\text {ith block }}, 0, \ldots, 0, \underbrace{a_{j i} \alpha_{i}}_{j \text { th block }}, 0, \ldots, 0), \\
{\left[D \Phi_{i j}^{1}\right] } & =(0, \ldots, 0, \underbrace{0}_{i \text { th block }}, 0, \ldots, 0, \underbrace{\alpha_{i}}_{j \text { th block }}, 0, \ldots, 0), \\
{\left[D \Phi_{i j}^{2}\right] } & =(0, \ldots, 0, \underbrace{0}_{i \text { th block }}, 0, \ldots, 0, \underbrace{\alpha_{i}}_{j \text { th block }}, 0, \ldots, 0) .
\end{aligned}
$$

## The infinitesimal deformation space

- Suppose that $p$ is a point of $\Phi_{P}^{-1}(0)$.
- Then the Zariski tangent space at $p$ is the kernel of the Jacobian matrix $D$.
- We call this the infinitesimal deformation space of $P$ at $p$
- If $4 f-N>0$ and $D$ has full rank, i.e. rank $D=\min (4 f, N)$, then $\mathfrak{D}_{P}(\hat{P})$ is locally a smooth manifold of dimension $4 f-N$.
- So if $p$ is the hyperbolic point, the hyperbolic structure on a Coxeter orbifold $\hat{P}$ deforms to a real projective structure which is not a hyperbolic structure.
- If $4 f-N \leq 0$ and $D$ has full rank, then $p$ is a isolated point in $\mathfrak{D}_{P}(\hat{P})$.
- So if $p$ is the hyperbolic point, the hyperbolic structure on $\hat{P}$ is projectively rigid in $\mathfrak{D}_{P}(\hat{P})$.
- Note that $4 f-N=4 f-\left(f+e+e_{2}\right)=3 f-e-e_{2}$.


## Main results

- (Theorem 1) Let $P$ be an ideal hyperbolic polyhedron whose dihedral angles are all equal to $\pi / 3$, and not a tetrahedron. Then $\mathfrak{D}_{P}(\hat{P})$ is locally a 6 -dimensional smooth manifold.
- (Proof) Writing hyperbolic equations in terms of the reflection vectors $b_{i}=2 \nu_{i}$ gives $n=f+e$ equations:

$$
\Psi_{i i}=\left\langle b_{i}, b_{i}\right\rangle-4=0, \Psi_{i j}=\left\langle b_{i}, b_{j}\right\rangle+4 \cos \left(\pi / n_{i j}\right)=0
$$

- Combining these gives a function $\Psi: \mathbb{R}^{4 f} \rightarrow \mathbb{R}^{n}$.
- Now consider the ker $D \Psi$ at a hyperbolic point $t$.
- The rows of the $n \times 4 f$ Jacobian matrix $\hat{D}=[D \Psi]$ are made up of blocks, each consisting of four entries:

$$
\left[D \Psi_{i i}\right]=(0, \ldots, 0, \underbrace{4 \alpha_{i}}_{i \text { th block }}, 0, \ldots, 0)
$$

$$
\left[D \Psi_{i j}\right]=(0, \ldots, 0, \underbrace{2 \alpha_{j}}_{i \text { th block }}, 0, \ldots, 0, \underbrace{2 \alpha_{i}}_{j \text { th block }}, 0, \ldots, 0) .
$$

## Main results

- $P$ contains no edges of order $2 \Longrightarrow N=n$.
- Each row of $D$ is a non-zero scalar multiple of a row of $\hat{D}$, so the ranks of $D$ and $\hat{D}$ are equal.
- Garland-Weil infinitesimal rigidity $\Longrightarrow D \Phi$ has full rank. $\square$
- This argument extends to convex hyperbolic polyhedra with trivalent but possibly hyperinfinite vertices, provided all edges have order at least 3.
- (Theorem 2) Consider the compact hyperbolic cubes such that each dihedral angle is $\pi / 2$ or $\pi / 3$. Up to symmetries, there exist 34 cubes. For the corresponding Coxeter orbifolds, 10 are deformable and the remaining 24 are projectively rigid.
- (Theorem 3) Consider the compact hyperbolic dodecahedra such that each dihedral angle is $\pi / 2$ or $\pi / 3$, and each face has at most two dihedral angles equal to $\pi / 2$. Up to symmetries, there exist 13 dodecahedra. For the corresponding Coxeter orbifolds, only 1 is deformable and the remaining 12 are rigid.


## The Main Algorithm

1. Tabulate compact hyperbolic cubes (or dodecahedra).
2. Apply the linear test of rigidity by hand.
3. Construct compact hyperbolic Coxeter cubes (or dodecahedra) obtained in step 1, by using Mathematica to solve the hyperbolic equations.
4. We compute the dimension of the Zariski tangent space for the hyperbolic point
5. If $D$ is of full rank, then the dimension of the space of infinitesimal deformations is the same as the dimension of the space of local deformations.
6. If $D$ is rank-deficient, we attempt to obtain the Gröbner basis of the ideal $\mathcal{I}$ generated by $\left\{\Phi_{k}=0\right\}_{k=1}^{N}$ with respect to a lexicographic order on the variables.

## A linear test for rigidity

- Let $P$ be a 3-dimensional Coxeter polyhedron in $\mathbb{S}^{3}$.
- Then there is a simple method to show the rigidity of the corresponding orbifold $\hat{P}$.

1. Find all the faces having more than two edges of order 2. We call them the rigid faces at level 1.
2. Relabel all edges of rigid faces of level 1 to become edges of order 2.
3. Again, find all other faces having more than two edges of order 2. We call them rigid faces at level 2. Relabel all edges of these faces to become edges of order 2.
4. Continue the process this manner.
5. If every face of $P$ occurs as a rigid face at level $k$ for some $k \geq 1$, then $\hat{P}$ is projectively rigid.
6. Once finished we recover the edge orders.

## Notations for figures and tables

- Each $e_{i}$ is an edge order, corresponding to a dihedral angle $\pi / e_{i}$,
- $\mathrm{O}=$ the number of variables - the number of Vinberg equations,
- $\mathrm{I}=$ the dimension of infinitesimal deformation space of real projective structures,
- A $=$ the dimension of local deformation space of real projective structures,
- $L=$ Is it possible to apply the linear test of rigidity? (yes or no) and the maximum level needed,
- $J=$ Does the calculation of the Jacobian $D$ give a full description of the local deformation space? (yes or no),
- $\mathrm{S}=$ the minimum of the singular values of the Jacobian $D$.


## The results for cubes

- Let $P$ be a compact hyperbolic cube, all of whose dihedral angles are $\pi / 2$ or $\pi / 3$.
- By step 1, the total number of such cubes is 34 .

- By step 2, we find that seventeen cubes are projectively rigid.
- Using steps 3-5, the exact algebraic computations of the dimensions of Zariski tangent spaces determine the dimensions of local deformation spaces for the eight cubes.
- Using step 6, we instead obtain the Gröbner basis of the ideal of Vinberg's equations by Mathematica.


## The list of cubes

| name | $e_{1} e_{2} \cdots e_{11} e_{12}$ | O | I | A | L | J |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| cu1 | 232222232223 | -3 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu2 | 232222232233 | -2 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu3 | 232222232322 | -3 | 0 | 0 | yes, level 1 | $\cdot$ |
| cu4 | 232222232323 | -2 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu5 | 232222232333 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu6 | 232222233322 | -2 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu7 | 232222233332 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu8 | 232222322223 | -3 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu9 | 232222322332 | -2 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu10 | 232222323223 | -2 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu11 | 232222323322 | -2 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu12 | 232222323323 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu13 | 232222323332 | -1 | 0 | 0 | yes, level 2 | $\cdot$ |
| cu14 | 232222333322 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu15 | 232222333332 | 0 | 0 | 0 | no | yes |
| cu16 | 232223233322 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu17 | 232223322323 | -1 | 1 | 1 | no | no |

## The list of cubes

| name | $e_{1} e_{2} \cdots e_{11} e_{12}$ | O | I | A | L | J |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| cu18 | 232223323323 | 0 | 1 | 1 | no | no |
| cu19 | 232223333322 | 0 | 0 | 0 | no | yes |
| cu20 | 232232232233 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu21 | 232232232323 | -1 | 1 | 1 | no | no |
| cu22 | 232232232333 | 0 | 1 | 1 | no | no |
| cu23 | 232232332322 | -1 | 0 | 0 | yes, level 3 | $\cdot$ |
| cu24 | 232232332323 | 0 | 0 | 0 | no | yes |
| cu25 | 232232332332 | 0 | 0 | 0 | no | yes |
| cu26 | 232233332223 | 0 | 1 | 0 | no | no |
| cu27 | 232233332323 | 1 | 2 | 1 | no | no |
| cu28 | 232322232233 | -1 | 0 | 0 | no | yes |
| cu29 | 232323232323 | 0 | 1 | 0 | no | no |
| cu30 | 232323323323 | 1 | 1 | 1 | no | yes |
| cu31 | 232323332323 | 1 | 1 | 1 | no | yes |
| cu32 | 232323333322 | 1 | 1 | 1 | no | yes |
| cu33 | 232333332323 | 2 | 3 | 2 | no | no |
| cu34 | 233223233322 | 0 | 1 | 1 | no | no |

## An example: cu21



Figure: cu21

- First, the set of all rigid faces is $\left\{F_{2}, F_{3}\right\}$, and the level of these faces is 1 .
- Second, we find the unit normals $\nu_{i}$ of cu21 as follows:

$$
\nu_{1}=(0,1,0,0), \quad \nu_{2}=(0,0,1,0), \quad \nu_{3}=\left(0,0,-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)
$$

## An example: cu21

$$
\nu_{4}=\left(x,-\frac{1}{2}, u, \frac{u}{\sqrt{3}}\right), \nu_{5}=\left(y,-\frac{1}{2}, 0, \frac{2 v}{\sqrt{3}}\right), \nu_{6}=(z, w, 0,0)
$$

- Solve the remaining six hyperbolic equations

$$
\begin{aligned}
& \left\langle\nu_{4}, \nu_{5}\right\rangle=0 \text { and }\left\langle\nu_{4}, \nu_{6}\right\rangle=\left\langle\nu_{5}, \nu_{6}\right\rangle=-1 / 2, \\
& \left\langle\nu_{4}, \nu_{4}\right\rangle=\left\langle\nu_{5}, \nu_{5}\right\rangle=\left\langle\nu_{6}, \nu_{6}\right\rangle=1 .
\end{aligned}
$$

- Third, using these $\alpha_{i}=J \nu_{i}$ we make the $25 \times 24$ Jacobian matrix $D$ of Vinberg's equations at the hyperbolic point.
- The rank of $D$ is 23 , and so $D$ is rank-deficient.
- The dimension of infinitesimal deformations of real projective structures is 1 .


## An example: cu21

- Finally, to obtain the dimension of local deformations of cu21, we compute Gröbner basis of the ideal $\left\langle\Phi_{1}, \ldots, \Phi_{N}\right\rangle$ with $N=f+e+e_{2}=25$.
- We introduce new coordinates on $\mathbb{R}^{24}$ by letting $c_{i}=b_{i}-t_{i}$.
- We compute a Gröbner basis of the ideal $\left\langle\Phi_{1}, \ldots, \Phi_{25}\right\rangle$ with respect to the lexicographic order with $c_{41}<c_{42}<c_{43}<c_{44}<$

$$
\begin{aligned}
& c_{51}<c_{52}<c_{53}<c_{54}<c_{61}<c_{62}<c_{63}<c_{64}<c_{11}<c_{12}<c_{13}< \\
& c_{14}<c_{21}<c_{22}<c_{23}<c_{24}<c_{31}<c_{32}<c_{33}<c_{34}
\end{aligned}
$$

- The Gröbner basis of cu21 is

$$
\begin{aligned}
& \left\{c_{34}, c_{33}, c_{32}, c_{31}, c_{24}, c_{23}, c_{22}, c_{21}, c_{14}, c_{13}, c_{12}, c_{64}, c_{63}\right. \\
& -c_{11}+\frac{2 c_{62}}{\sqrt{5}}+2 c_{11} c_{62}, \sqrt{5} c_{61}+3 c_{62}, c_{54}, c_{53},-c_{11}+\frac{2 c_{52}}{\sqrt{5}}+c_{11} c_{52} \\
& \left.-c_{52}+c_{62}+c_{52} c_{62}, \sqrt{5} c_{51}+c_{52}, c_{44}, c_{43}, c_{42}-c_{52}, \sqrt{5} c_{41}+c_{52}\right\}
\end{aligned}
$$

- The dimension of local deformations is also 1 .


## The results for dodecahedra

- Let $P$ be a compact hyperbolic dodecahedron of all of whose dihedral angles are $\pi / 2$ or $\pi / 3$.
- Assume that each face has less than three edges which are of order 2.
- The total number of such cubes is 13 up to symmetries.



## The list of Dodecahedra

| name | $e_{1} e_{2} \cdots e_{29} e_{30}$ | 0 | I | A | J | S |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| do1 | 232332323232323333332323233232 | -6 | 0 | 0 | yes | 0.17653 |
| do2 | 232332333232223333332332233233 | -5 | 0 | 0 | yes | 0.13121 |
| do3 | 232332333232223333332333233223 | -5 | 0 | 0 | yes | 0.14468 |
| do4 | 232332333232223333332333233232 | -5 | 0 | 0 | yes | 0.13707 |
| do5 | 232332333232323322332333233232 | -6 | 0 | 0 | yes | 0.18151 |
| do6 | 232332333232333322332332233233 | -5 | 0 | 0 | yes | 0.11944 |
| do7 | 232333223233323333322223333332 | -5 | 0 | 0 | yes | 0.12703 |
| do8 | 232333233232223333322332333232 | -6 | 0 | 0 | yes | 0.09580 |
| do9 | 232333233232223333322333333232 | -5 | 0 | 0 | yes | 0.09365 |
| do10 | 232333233233223333322233333322 | -5 | 0 | 0 | yes | 0.08277 |
| do11 | 232333333222223333333333232323 | -4 | 0 | 0 | yes | 0.06115 |
| do12 | 23233333322233232233233323323 | -6 | 0 | 0 | yes | 0.12412 |
| do13 | 232333333233223233323233323323 | -4 | 1 | 1 | no | $\cdot$ |

## The do13



- do13 has the five-fold rotational symmetry of the axis passing through the centers of top and bottom faces


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