On maximum matching width

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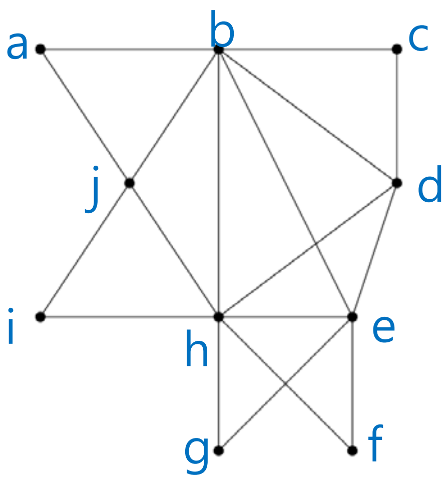
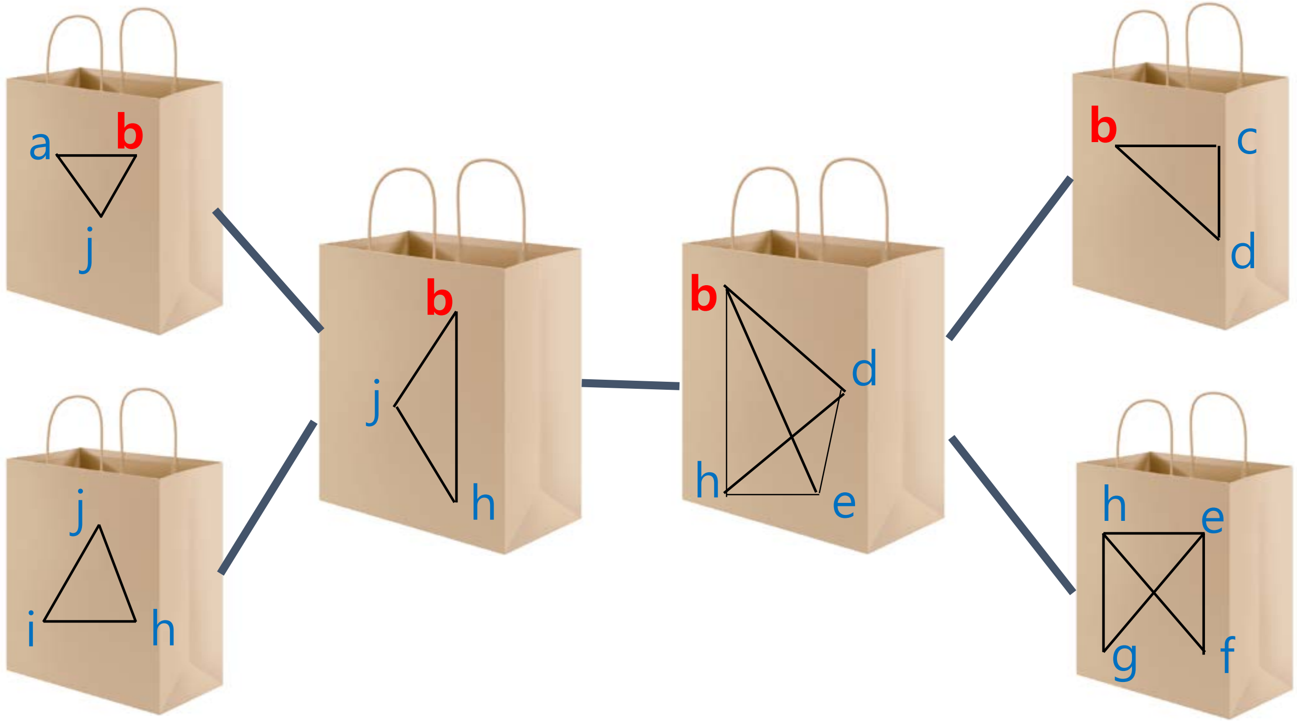
Graph width parameters

- tree-width (Halin 1976, Robertson and Seymour 1984)
- branch-width (Robertson and Seymour 1991)
- carving-width (Seymour and Thomas 1994)
- clique-width (Courcelle and Olariu 2000)
- rank-width (Oum and Seymour 2006)
- maximum matching-width (Vatshelle 2012)

A *tree-decomposition* of a graph G is a pair $(T, \{X_t\}_{t \in V(T)})$ consisting of a tree T and a family $\{X_t\}_{t \in V(T)}$ of subsets X_t of $V(G)$, called *bags*, satisfying the following three conditions:

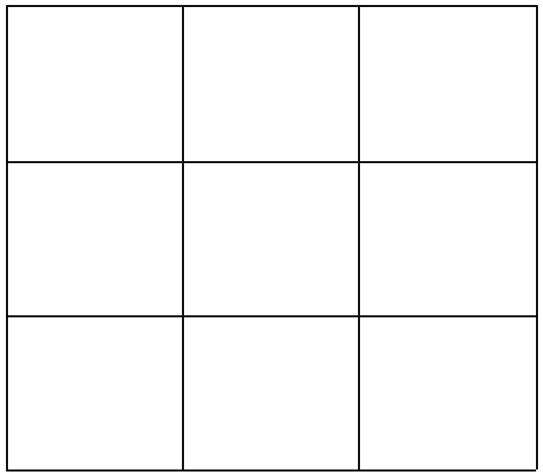
1. each vertex of G is in at least one bag,
2. for each edge uv of G, there exists a bag that contains both u and v,
3. **if X_i and X_j both contain a vertex v, then all bags X_k in the path between X_i and X_j contain v as well.**



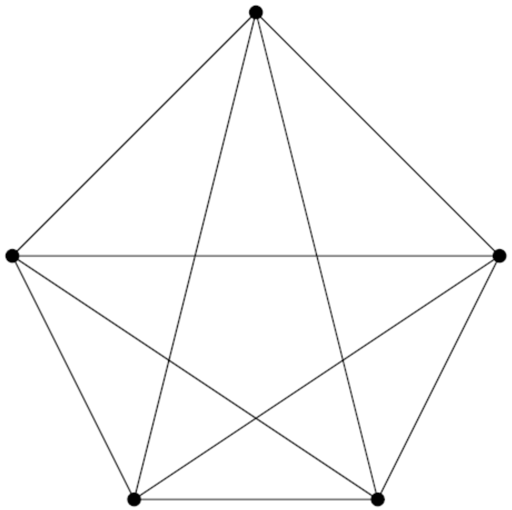


Examples

- tree-width $\leq 1 \Leftrightarrow$ a forest $\Leftrightarrow$ no cycle
- tree-width $\leq 2 \Leftrightarrow$ a series-parallel graph
  $\Leftrightarrow$ no K_4 minor
- The tree-width of a $k \times k$ grid is k.
- The tree-width of K_n is $n - 1$.



4×4 grid



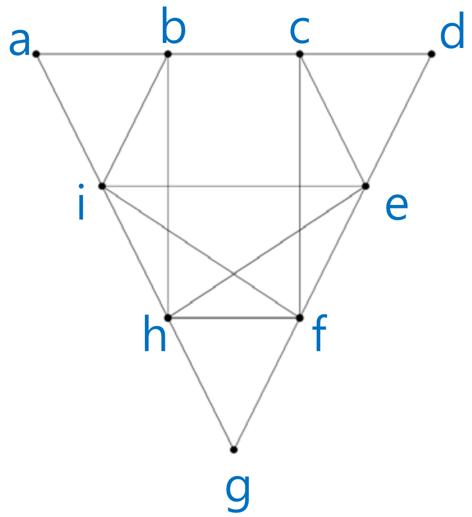
K_5

A *branch-decomposition* (T, L) over the vertices of a graph G consists of

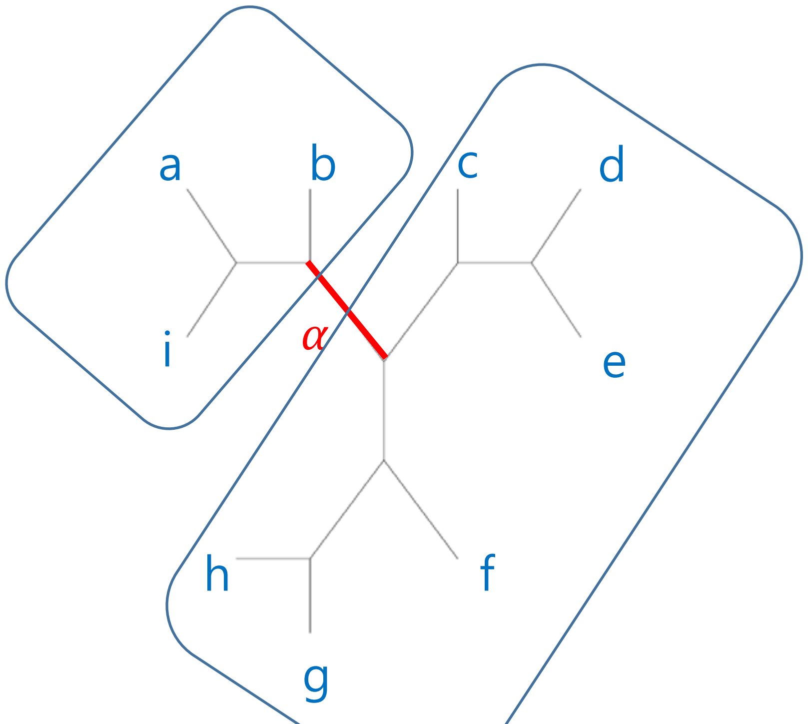
- a tree T where all internal vertices have degree 3 and
- a bijective function L from the leaves of T to the vertices of G.

The mm-value of an edge α of T is

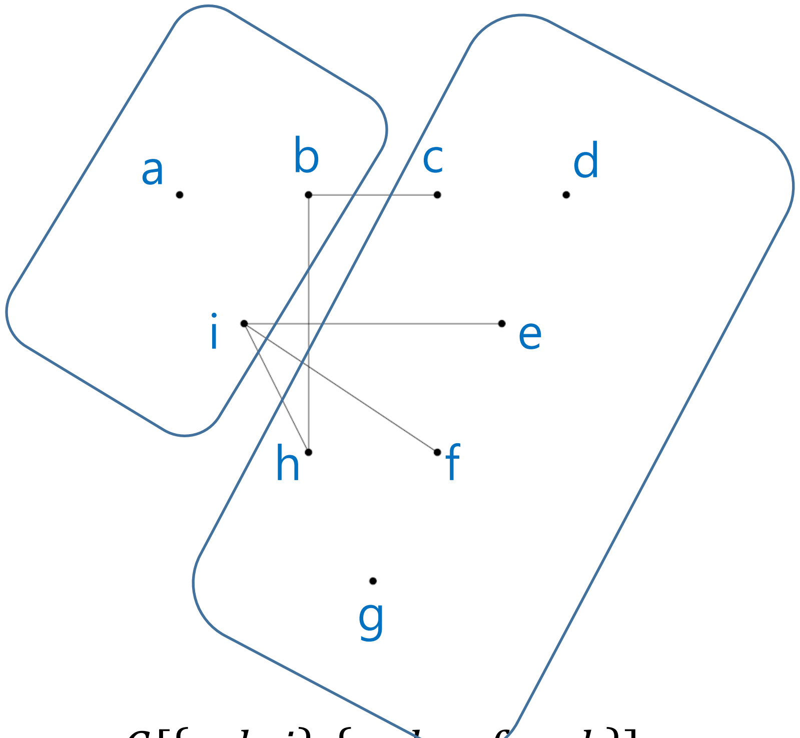
the size of the maximum matching of $G[\{a,b,i\},\{c,d,e,f,g,h\}]$.



G



(T, L)



$G[\{a,b,i\},\{c,d,e,f,g,h\}]$

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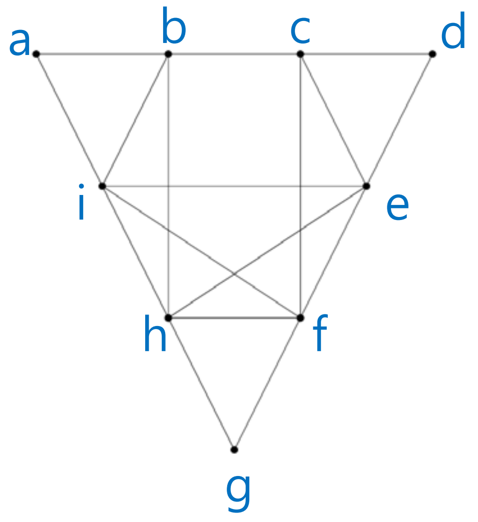

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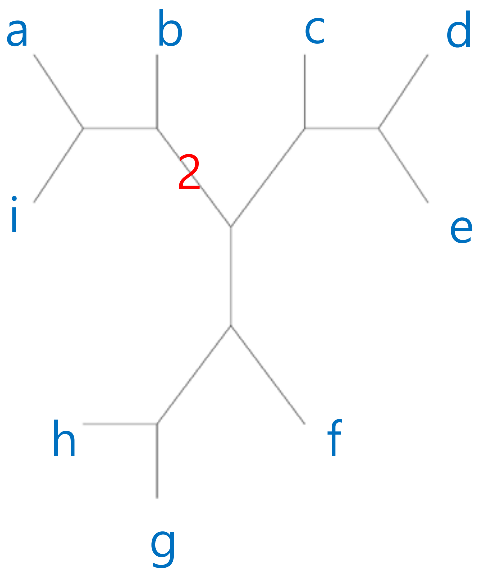
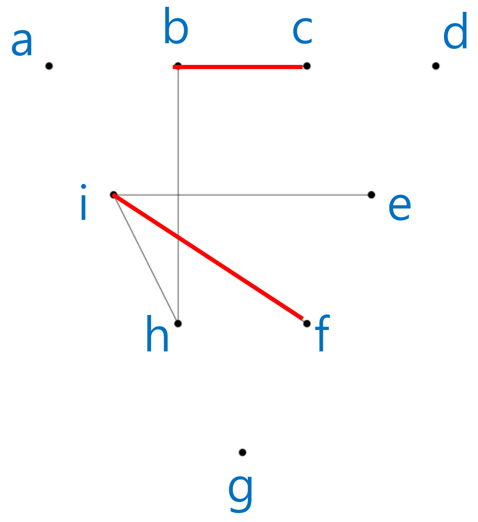


(T, L)



$G[\{a, b, i\}, \{c, d, e, f, g, h\}]$

A *branch-decomposition* (T, L) over the vertices of a graph G consists of

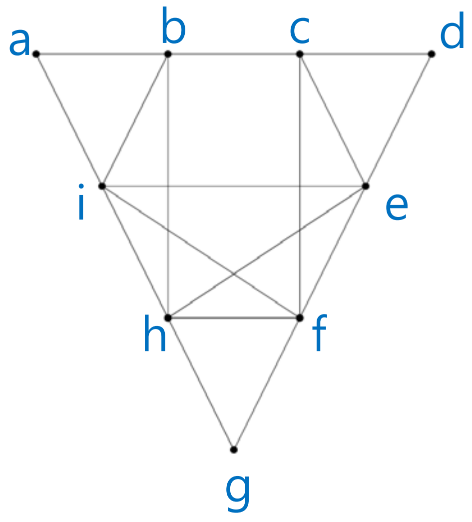
- a tree T where all internal vertices have degree 3 and
- a bijective function L from the leaves of T to the vertices of G.

The mm-width of a branch-decomposition (T, L) is

- the maximum mm-value among all edges.

The maximum matching-width (mm-width, $mmw(G)$) of a graph G is

- the minimum mm-width over all possible branch-decompositions over $V(G)$.



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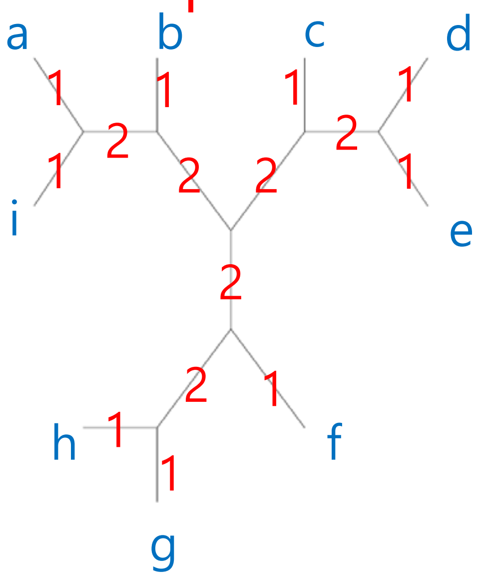


(T, L)

The mm-width of (T, L) is 2

Properties

Theorem (Vatshelle 2012)

For every graph G,

$$mmw(G) \leq tw(G) + 1 \leq 3\, mmw(G).$$

A graph G has bounded tree-width if and only if G has bounded mm-width.

Inequalities

Theorem (Vatshelle 2012)

For every graph G,

$$mmw(G) \leq \max(brw(G), 1) \leq tw(G) + 1 \leq 3\, mmw(G).$$

Theorem (Vatshelle 2012)

For every graph G, $rw(G) \leq mmw(G)$.

Algorithms

$$O^*(f(k,n)) = f(k,n)poly(n)$$

Theorem (Sæther, Telle 2014)

The cut-function mm is submodular.

Corollary (Oum 2009)

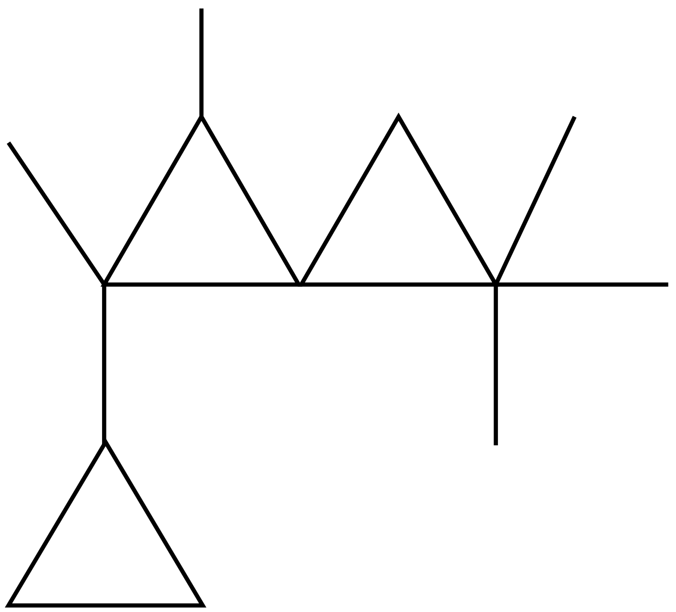
Given a graph G, we can compute a decomposition over $V(G)$ having optimal mm-width in time $O^*(2^{|V(G)|})$.

Corollary (Oum, Seymour 2006)

Given a graph G, a branch-decomposition over $V(G)$ of mm-width at most $3mmw(G) + 1$ can be found in time $O^*(2^{3mmw(G)})$.

Properties

- mm-width ≤ 1
  $\Leftrightarrow$ every maximal 2-connected subgraph is K_2 or K_3
  $\Leftrightarrow$ no C_4 minor



- The mm-width of K_n is $\left\lceil \frac{n}{3} \right\rceil$.

Questions

- Characterize a class of graphs having mm-width ≤ 2.

- What is the mm-width of a $k \times k$ grid?

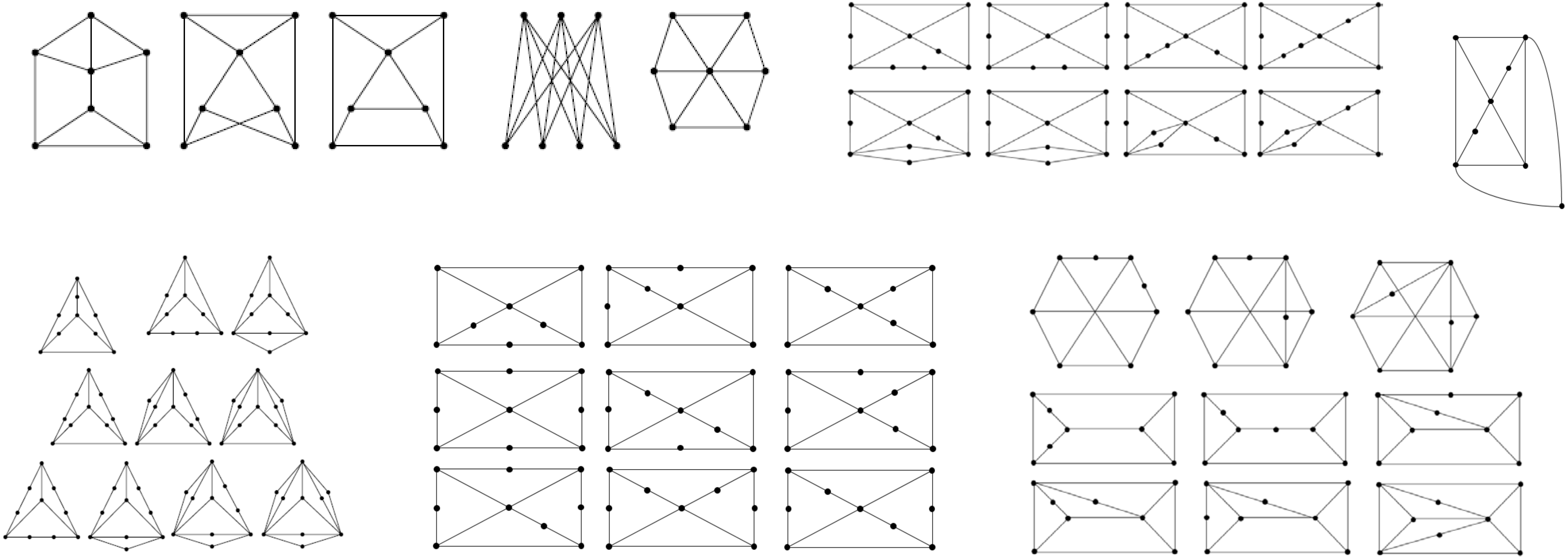
Questions

- Characterize a class of graphs having mm-width ≤ 2.
  `minor-closed'+ `well-quasi-ordering'
  $\Rightarrow$ It can be characterized by finite forbidden minors.

- What is the mm-width of a $k \times k$ grid?

Theorem (J., Ok, Suh 2015+)

There are 42 forbidden minors for mm-width at most 2.



Questions

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$$rw(G_k) \leq mmw(G_k) \leq brw(G_k)$$

Questions

- Characterize graphs having mm-width ≤ 2.

- What is the mm-width of a $k \times k$ grid G_k?

$$k - 1 = rw(G_k) \leq mmw(G_k) \leq brw(G_k) = k$$

Questions

- Characterize graphs having mm-width ≤ 2.

- What is the mm-width of a $k \times k$ grid G_k?

Theorem (J., Oum, Suh 2015+)

The mm-width of a $k \times k$-grid is k.

Why mm-width?

Theorem (Vatshelle 2012)

For every graph G,

$$mmw(G) \leq tw(G) + 1 \leq 3\, mmw(G).$$

A graph G has bounded tree-width if and only if G has bounded mm-width.

Why mm-width?

We want to solve Graph Problems efficiently.

A Dominating Set of a graph G is a set D of vertices such that $N(D) \cup D = V(G)$.

What is the minimum size of a dominating set of G?

Why mm-width?

Using tree-width

Theorem (van Rooij, Bodlaender, Rossmanith 2009)

Minimum Dominating Set Problem can be solved in time $O^*(3^t)$ when a graph and its tree-decomposition of width t is given.

Theorem (Lokshtanov, Marx, Saurabh 2011)

Minimum Dominating Set Problem cannot be solved in time $O^*((3-\varepsilon)^t)$ where t is the tree-width of the given graph unless the strong exponential time hypothesis fails.

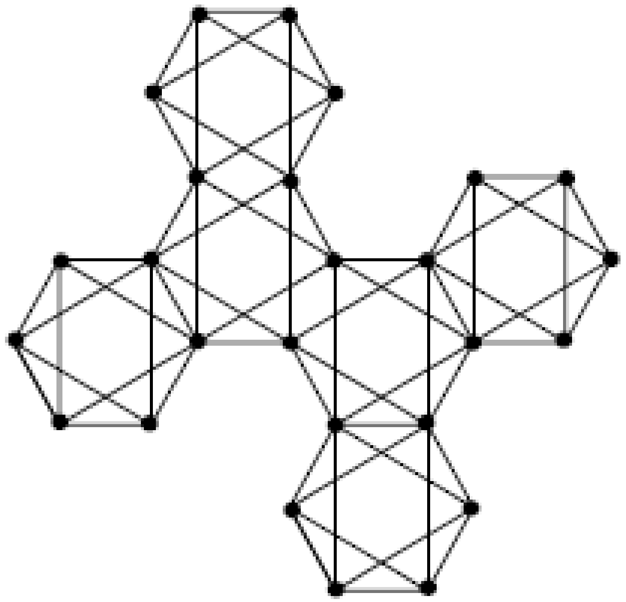
Why mm-width?

Using mm-width

Theorem (J., Sæther, Telle IPEC2015)

Minimum Dominating Set Problem can be solved in time $O^*(8^m)$ when a graph and its branch-decomposition of mm-width m is given.

Why mm-width?



Using tree-width: $O^*(3^t)$
Using mm-width: $O^*(8^m)$

Our algorithm is faster when $8^m < 3^t$, that is,

$$1.893\, mmw(G) < tw(G).$$

Note that for every graph G,

$$mmw(G) \leq tw(G) + 1 \leq 3\, mmw(G).$$

What if **only a graph** is an input?

Theorem (Oum, Seymour 2006)

Given a graph G, a branch-decomposition over $V(G)$ of mm-width at most $3mmw(G) + 1$ can be found in time $O^*(2^{3mmw(G)})$.

Runtime : $O^*(8^m) = O^*(8^{3mmw(G)})$

Our algorithm is faster if an input graph G satisfies $1.55\, mmw(G) < tw(G)$.

Theorem (Amir 2010)

Given a graph G, a tree-decomposition over $V(G)$ of width at most $3.67tw(G)$ can be found in time $O^*(2^{3.7tw(G)})$.

Runtime : $O^*(3^t) = O^*(3^{3.67tw(G)})$

Why mm-width?

Using mm-width

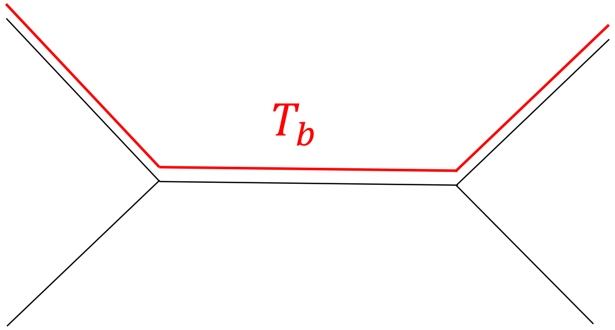
Theorem (J., Sæther, Telle 2015)

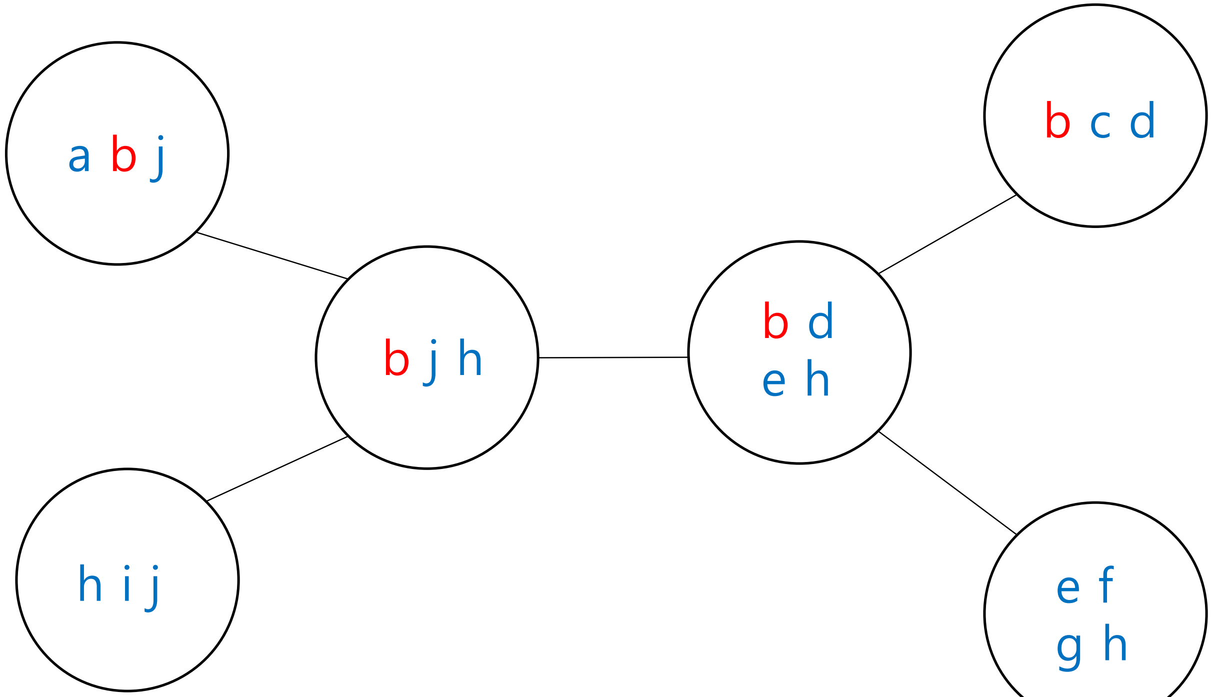
Minimum Dominating Set Problem can be solved in time $O^*(8^m)$ when a graph and its branch-decomposition of mm-width m is given.

Proof ideas

1. New characterization of graphs of mm-width at most k
2. Dynamic programming
3. Fast Subset Convolution, Monotonicity

New characterization (tree-width)





tree-decomposition

New characterization (tree-width)

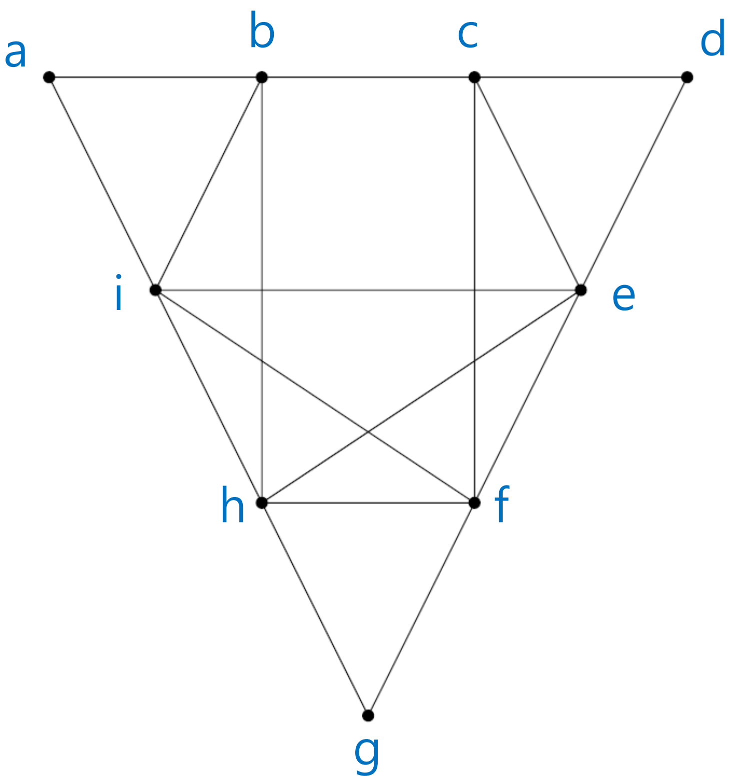
For any $k \geq 2$, a graph G on vertices $v_1, v_2, \dots, v_n$ has tree-width at most k *if and only if*

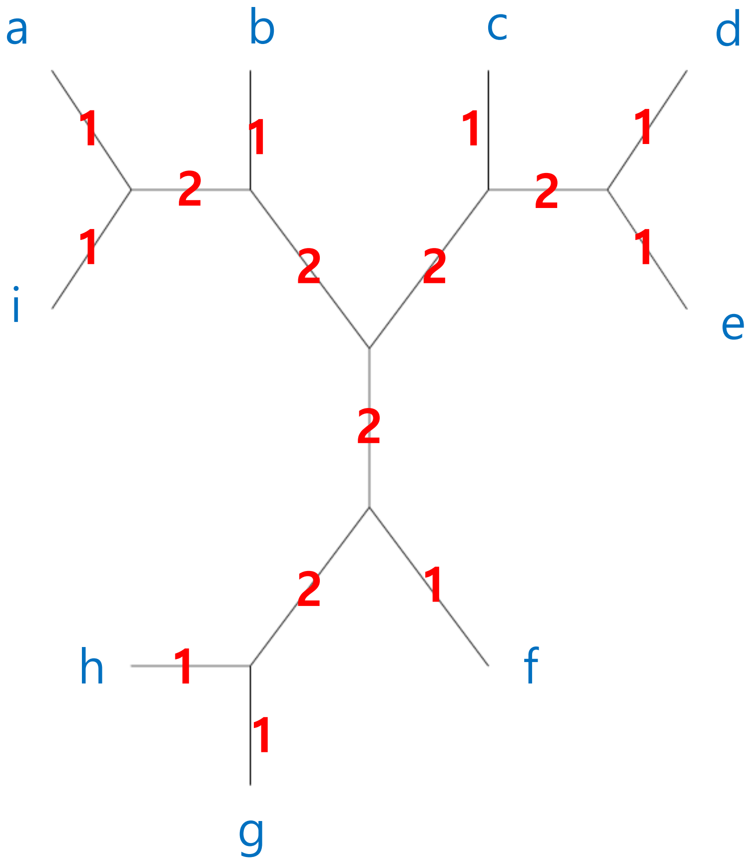
there are subtrees $T_1, T_2, \dots, T_n$ of a tree T where all internal vertices have degree 3

such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one vertex of T in common,

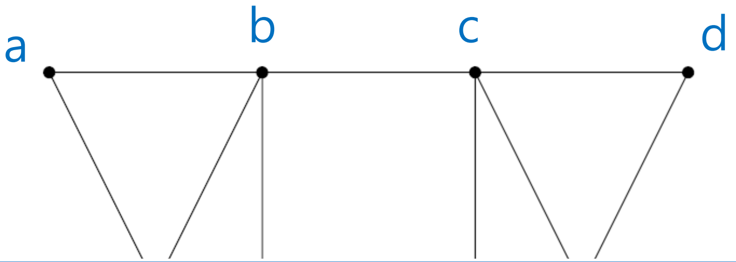
2) for each vertex of T, there are at most $k - 1$ subtrees containing it.

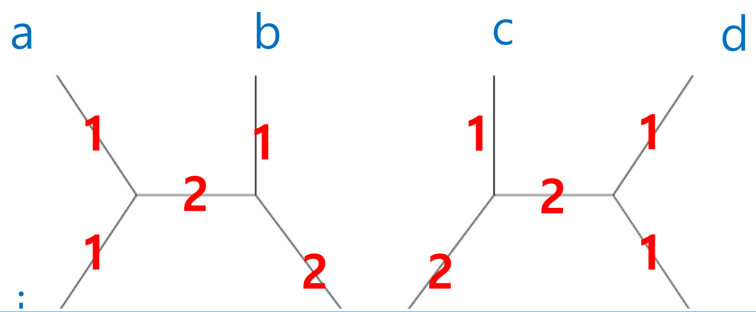
New characterization (mm-width)





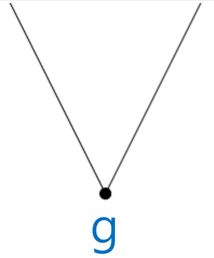
New characterization (mm-width)

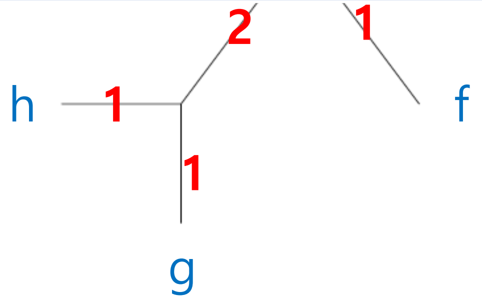




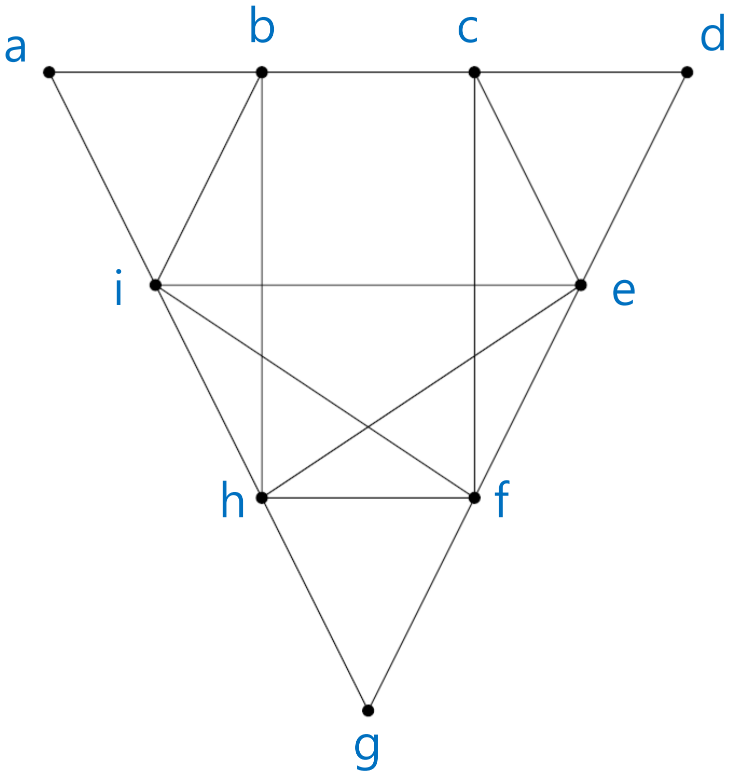
Theorem (König 1931)

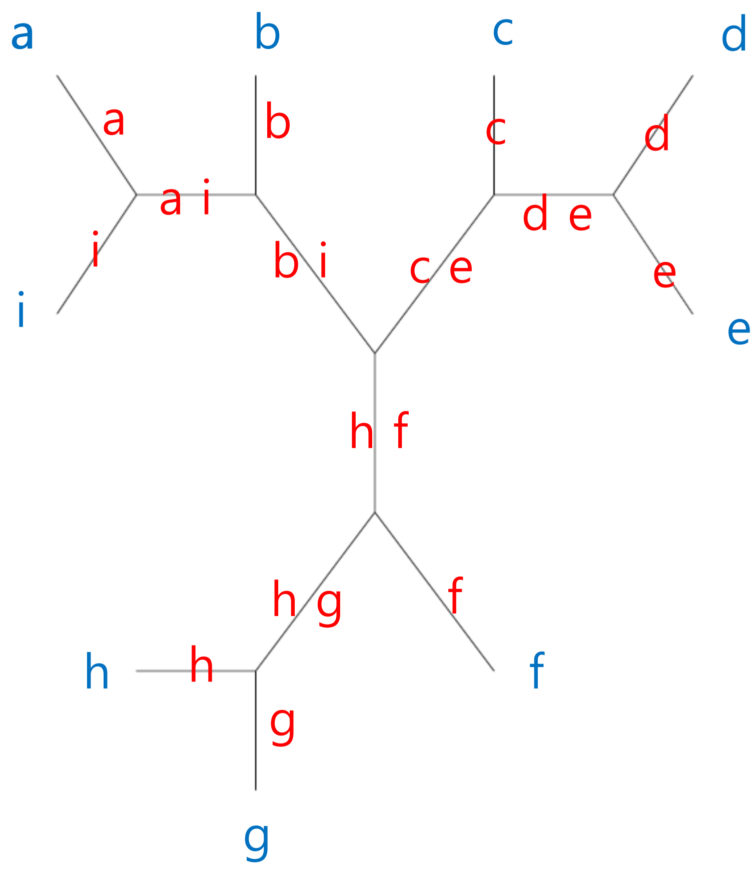
For every bipartite graph G,
the size of a maximum matching is equal to the size of a minimum vertex cover.





New characterization (mm-width)





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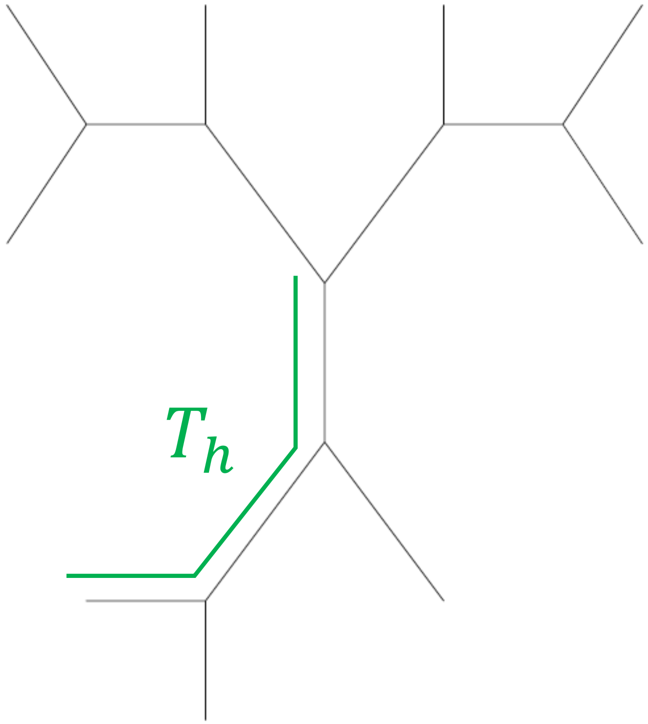

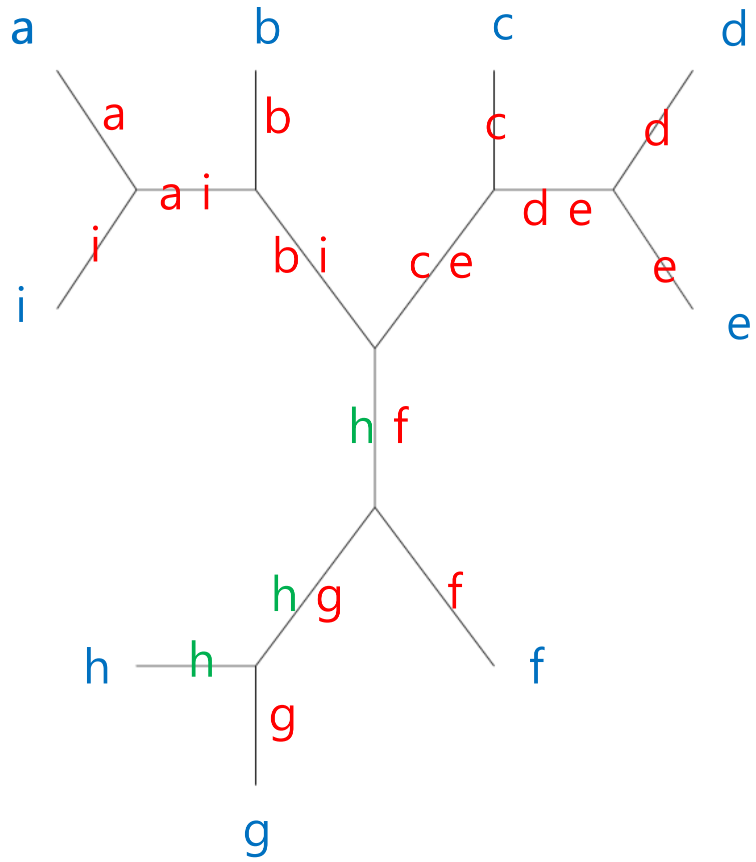

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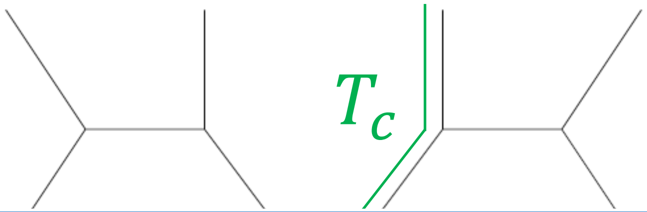
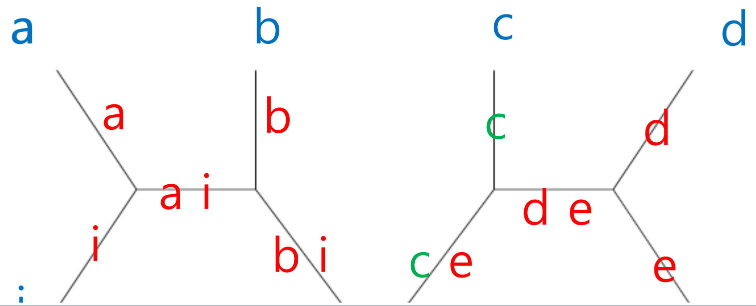
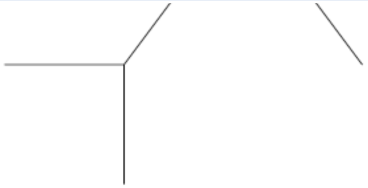


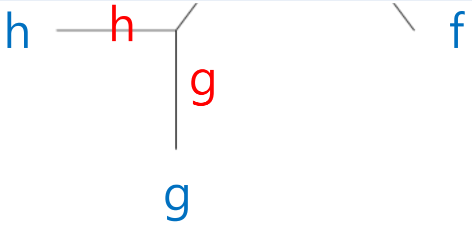


Key Lemma

A vertex cover C_K is the A-König cover of a bipartite graph $G = (A \cup B, E)$
if and only if for each minimum vertex cover C' of G we have

$$A \cap C' \subseteq A \cap C_K, \qquad B \cap C' \supseteq B \cap C_K.$$





Theorem (J., Sæther, Telle 2015)

For any $k \geq 2$, a graph G on vertices $v_1, v_2, \dots, v_n$ has mm-width at most k *if and only if* there are subtrees $T_1, T_2, \dots, T_n$ of a tree T where all internal vertices have degree 3 such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one vertex of T in common,

2) for each edge of T, there are at most k subtrees containing it.

Theorem

A graph G has $tw(G) \leq k$ if and only if it is a subgraph of a chordal graph H such that the maximum size of a clique in H is at most k.

Theorem (J., Sæther, Telle 2015)

A graph G has $mmw(G) \leq k$ if and only if it is a subgraph of a chordal graph H and for every maximal clique X of H there exists $A, B, C \subseteq X$ with $A \cup B \cup C = X$ and $|A|, |B|, |C| \leq k$ such that any subset of X that is a minimal separator of H is a subset of either A, B, or C.

New characterization

For any $k \geq 2$, a graph G on vertices $v_1, v_2, \ldots, v_n$ has
tree-width (mm-width) at most k *if and only if*
there are subtrees $T_1, T_2, \ldots, T_n$ of a tree T where all internal
vertices have degree 3
such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one
vertex of T in common,
2) for each vertex (edge) of T, there are
at most $k-1$ (at most k) subtrees containing it.

Thank you

New characterization

For any $k \geq 2$, a graph G on vertices $v_1, v_2, \dots, v_n$ has
tree-width (mm-width, branch-width) at most k *if and only if*
there are subtrees $T_1, T_2, \dots, T_n$ of a tree T where all internal vertices have degree 3
such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one vertex (vertex, edge) of T in common,
2) for each vertex (edge, edge) of T, there are at most $k-1$ (at most k, at most k) subtrees containing it.

Paul, Telle, Edge-maximal graphs of branchwidth k: the k-branches. 2009